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EDITED BY

ORMOND STONE W. E. BYERLY H. S. WHITE
F. S. WOODS MAXIME BÔCHER
E. V. HUNTINGTON

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NOTE ON SERIES OF ANALYTIC FUNCTIONS*

By Cesare Arzelà

I.

In the "Note on the functions defined by infinite series whose terms are analytic functions of a complex variable, etc.,†" Professor Osgood has demonstrated the following theorem, the generality and importance of which are obvious.

"*If the infinite series*

$$f_1(z) + f_2(z) + \cdots \tag{1}$$

whose terms are all functions of z, single valued and analytic throughout a region T of the complex z-plane, converges for all values of z pertaining to a set of points which is everywhere dense throughout T (and which, in particular, may be enumerable); and if, furthermore, the relation

$$|f_1(z) + \cdots + f_n(z)| \leqq G$$

holds for all points z of this set (and hence of T), and for all values of n, G being a positive constant, then the series converges for all values of z in T and the function $F(z)$ defined by the series:

$$F(z) = f_1(z) + f_2(z) + \cdots$$

is analytic throughout T."

The simplicity of the following demonstration, which depends on certain propositions already obtained by the author, may justify its publication.

II.

The propositions in question are the following:

If there be in a region T of the x, y plane an infinite set of functions

$$(a) \qquad u_1(x, y),\ u_2(x, y),\ \cdots,\ u_s(x, y),\ \cdots,$$

each continuous and included between the fixed numbers l and L then:

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* The editors acknowledge their indebtedness to Dr. C. N. Haskins, who has kindly performed the task of translating this article from the Italian manuscript, and of adding in §VII a translation of an extract from one of the author's earlier papers.

† Osgood, Annals of Mathematics, ser. 2, vol. 3 (1901–02), p. 26.

1. The necessary and sufficient condition that there exist at least one continuous limiting function for this set is that the set contain an infinite number of functions which are *equably continuous*.

2. A set of functions taken from the set a is said to be *equably continuous* in T if, given an arbitrarily small positive number σ, a positive number δ can always be found such that in every part of T whose maximum chord does not exceed δ the oscillation of each of the functions in question is less than σ.

3. A *sufficient* condition for the equable continuity of such a set of functions is that the two difference quotients

$$\frac{u(x_1, y) - u(x_2, y)}{x_1 - x_2}, \qquad \frac{u(x, y_1) - u(x, y_2)}{y_1 - y_2}$$

remain between two finite numbers l and L, for all functions $u(x, y)$ of the set, and for all points (x_1, y), (x_2, y), (x, y_1), (z, y_2) possible in T.

4. If the partial derivatives

$$\frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial y}$$

exist, it is sufficient that in T, including the contour, they lie between fixed numbers.

These propositions are demonstrated by methods similar to those used by the author in an earlier paper for the corresponding propositions in the theory of functions of a single variable.*

III.

Let us recall further from the theory of analytic functions of a single variable the propositions :—

1. Given an ordered set,

$$\phi_1(z), \quad \phi_2(z), \cdots, \quad \phi_s(z), \cdots$$

of single valued analytic functions uniformly convergent in T to a function $f(z)$, then this function is in T a single valued analytic function.

2. The series of derivatives

$$\phi_1'(z), \quad \phi_2'(z), \cdots, \quad \phi_s'(z), \cdots$$

converges uniformly to $f'(z)$.

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* See §VII where those portions of this paper with which we are here concerned are reproduced.

IV.

The above being assumed, let us consider a succession of functions single valued and analytic in T:

$$\phi_1(z), \quad \phi_2(z), \cdots, \quad \phi_s(z), \cdots, \tag{1}$$

for each of which there exists at least one point z (which may vary from function to function) in which

$$|\phi_s(z)| < M, \tag{2}$$

M being a fixed number (the same for every z in T) and

$$|\phi_s'(z)| < L, \tag{3}$$

L being likewise a fixed number. Then there exists at least one single valued analytic function $S(z)$ to which *one* succession of functions taken from (1) converges uniformly.

If we put $z = x + yi$, $\phi_s(z) = u_s(x, y) + iv_s(x, y)$ then from (2) follows that, in some point (x, y), we have

$$\begin{aligned} |u_s(x, y)| &< M, \\ |v_s(x, y)| &< M, \end{aligned} \tag{4}$$

whatever be s. The point (x, y), however, may vary with s. From

$$\phi_s'(z) = \frac{\partial u_s}{\partial x} + i\frac{\partial v_s}{\partial x} = \frac{\partial v_s}{\partial y} - i\frac{\partial u_s}{\partial y}$$

follows that the condition (3) is equivalent to the following:

$$\begin{aligned} \left|\frac{\partial u_s}{\partial x}\right| < L, \qquad \left|\frac{\partial u_s}{\partial y}\right| < L, \\ \left|\frac{\partial v_s}{\partial x}\right| < L, \qquad \left|\frac{\partial v_s}{\partial y}\right| < L. \end{aligned} \tag{5}$$

This last condition gives, by means of §II, (4), that the successions

$$u_1(x, y), \quad u_2(x, y), \cdots, \quad u_s(x, y), \cdots, \tag{6}$$

$$v_1(x, y), \quad v_2(x, y), \cdots, \quad v_s(x, y), \cdots \tag{7}$$

are equably continuous.

For, from the identity

$$u(x+h, y+k) - u(x, y) = \frac{u(x+h, y+k) - u(x+h, y)}{k}\, k + u\,\frac{(x+h, y) - u(x, y)}{h}\, h$$

(in which (x, y) may be for each of the functions the point in which the relations (4) are true, and $(x+h, y+k)$ any point whatever in T), and by means of (5), we see that all the $u_s(x, y)$ remain always in absolute value less than a fixed number*

$$M + pLd,$$

where d is the longest chord which can be drawn in T, and p is a fixed integer. The same holds for the $v_s(x, y)$. The succession of functions (6) satisfies the conditions of proposition (2) §II. There is, therefore, for this succession at least one continuous limiting function $u(x, y)$.

The particular succession of functions converging uniformly to $u(x, y)$ may for example be chosen by taking from (6) those functions of smallest index, s, satisfying the inequalities

$$|u(x, y) - u_{s_1}(x, y)| < \epsilon,$$

$$|u(x, y) - u_{s_2}(x, y)| < \frac{\epsilon}{2},$$

ϵ being a small positive number which may be taken at pleasure.

Under this assumption, the succession

$$u_{s_1}(x, y), \quad u_{s_2}(x, y), \cdots \tag{8}$$

converges uniformly to $u(x, y)$ throughout T. The succession of associated functions

$$v_{s_1}(x, y), \quad v_{s_2}(x, y), \cdots \tag{9}$$

will also admit continuous limiting functions. We cannot say at once that it will admit but one. Let, however, $v'(x, y)$, $v''(x, y)$ be two such functions. Then we shall find that their difference is a constant. For let $v'(x, y)$ be the limiting function to which the succession

$$v_{s_{n_1}}(x, y), \quad v_{s_{n_2}}(x, y), \cdots \tag{9'}$$

* By means of lines parallel to the axes and wholly contained in T, in number fewer than a fixed number p, we may always join the point (x, y) to the point $(x+h, y+k)$.

taken from (9) converges uniformly, and similarly let $v''(x, y)$ be the function to which the succession

$$(9'') \qquad v_{s_{m_1}}(x, y), \quad v_{s_{m_2}}(x, y), \cdots$$

converges uniformly. Then if we consider the succession of functions associated with (9′) and (9″), *i. e.*

$$u_{s_{n_1}}(x, y), \quad u_{s_{n_2}}(x, y), \cdots$$

and

$$u_{s_{m_1}}(x, y), \quad u_{s_{m_2}}(x, y), \cdots,$$

these two, forming a part of (8), converge to a single function $u(x, y)$. But the two successions of analytic functions

$$u_{s_{m_1}}(x, y) + iv_{s_{n_1}}(x, y), \quad u_{s_{n_2}}(x, y) + iv_{s_{n_2}}(x, y) \cdots$$

and

$$u_{s_{m_1}}(x, y) + iv_{s_{m_1}}(x, y), \quad u_{s_{m_2}}(x, y) + iv_{s_{m_2}}(x, y) \cdots$$

converge uniformly to the functions

$$u(x, y) + iv'(x, y), \quad u(x, y) + iv''(x, y)$$

respectively, which will be, therefore, single valued and analytic in T. From this it follows that $v'(x, y)$ and $v''(x, y)$, as asserted, differ at most by a constant. The proposition is proved.

V.

Consider now Osgood's hypothesis.

1. *Let* $f_1(z)$, $f_2(z)$, . . . *be analytic functions in* T, *and, letting*

$$s_n(z) = f_1(z) + f_2(z) + \cdots + f_n(z),$$

suppose we have for every n, *and for every* z *in* T,

$$|s_n(z)| < L,$$

where L is fixed.

For a moment let us suppose T simply connected. Then consider the series

$$\int_{z_0}^{z} f_1(z)dz + \int_{z_0}^{z} f_2(z)dz + \cdots,$$

and let

$$S_n(z) = \int_{z_0}^{z} f_1(z)dz + \int_{z_0}^{z} f_2(z)dz + \cdots + \int_{z_0}^{z} f_n(z)dz,$$

z_0 being an arbitrary fixed point and z any variable point whatever.

The path of integration can always be considered as of length inferior to a given number H. It follows that

$$|S_n(z)| < HL,$$

for every n and for every z. But further

$$S'_n(z) = f_1(z) + f_2(z) + \cdots + f_n(z) = s_n(z),$$

and consequently, by hypothesis, for every n and for every z,

$$|S'_n(z)| < L.$$

Consider now the succession

$$S_1(z),\ S_2(z),\ \cdots,\ S_n(z),\ \cdots.$$

There must exist one or more single valued analytic functions to each one of which converge uniformly successions of functions obtained from the given one. If $\Sigma(z)$ is one of these limiting functions, for example the one to which the succession

$$S_{n_1}(z),\quad S_{n_2}(z),\ \cdots$$

converges, then the succession of the derivatives,

$$S'_{n_1}(z),\quad S'_{n_2}(z),\ \cdots,$$

will admit for limiting function the derivative $\Sigma'(z)$ which will be, moreover, an analytic function.

2. Let us now adjoin the other hypothesis, viz., *that the given series*

$$(a) \qquad f_1(z) + f_2(z) \cdots$$

shall be convergent in the points of a set everywhere dense in T. From this follows that any succession taken from the above, as

$$S'_1(z),\quad S'_2(z),\ \cdots,$$

converges always to a single limiting function.

For if $\Sigma'_1(z)$, $\Sigma'_2(z)$ are two such limiting functions, viz. those of the partial successions

$$S'_{m_1}(z),\quad S'_{m_2}(z),\ \cdots,\quad S'_{m_s}(z),\ \cdots,$$

$$S'_{n_1}(z),\quad S'_{n_2}(z),\ \cdots,\quad S'_{n_s}(z),\ \cdots,$$

respectively, then in all points z of the set in question they have the same limit, viz., that of the proposed series a; therefore, in all points of the every-

where dense set of points the two functions $\Sigma_1'(z) = \Sigma_2'(z)$ coincide. They are continuous in those points, so that we have in all points of T

$$\Sigma_1'(z) = \Sigma_2'(z).$$

From this we have

$$\Sigma_1(z) - \Sigma_2(z) = C = \text{constant},$$

and since, in the point z_0, all of the integrals

$$\int_{z_0}^{z} f_1(z)dz\,, \quad \int_{z_0}^{z} f_2(z)dz, \ldots$$

are zero, it follows that $C = 0$ and $\Sigma_1(z) = \Sigma_2(z)$.

We have, therefore, proved that, under the assumed hypothesis, the functions $S_n(z)$ converge with increasing n uniformly in T to a single valued analytic function

$$S(z) = \int_{z_0}^{z} f_1(z)dz + \int_{z_0}^{z} f_2(z)dz + \cdots.$$

But then the derivative of this function is also a single valued analytic function

$$S'(z) = f_1(z) + f_2(z) + \cdots,$$

and that was precisely what we wished to prove.

VI.

We have supposed in the demonstration of the last theorem that the region T was simply connected, but this restriction can be easily removed.

The region T in which the series

$$f_1(z) + f_2(z) + \cdots$$

is given may be multiply connected. We reduce the problem by making it simply connected by introducing suitable cuts. Let T′ be the new region. By the preceding proposition the sum of the series

$$S(z) = f_1(z) + f_2(z) + \cdots$$

will be a single valued analytic function in T'. But we observe that the cuts may be deformed and that we may obtain from T another region T'', also simply connected, such that the original cuts fall wholly in T'', in which the $S(z)$ is by the same proposition a single valued analytic function.

$S(z)$ is therefore a single valued analytic function in those parts of T in which occur the cuts which form the boundary of T'; and therefore is likewise so in T.

VII.*

We shall now prove the propositions of which those of §II are the extension.

1. THEOREM. The necessary and sufficient conditions that a given succession of functions

$$(a) \qquad u_1(x), \quad u_2(x), \cdots,$$

defined in an interval $a \ldots b$, may converge to a limiting function is that, given an arbitrarily small positive number σ there can always be determined a corresponding integer m_σ such that for all values of x in the interval $a \ldots b$, and for all positive integers p,

$$|u_{m_\sigma}(x) - u_{m_\sigma+p}(x)| < \sigma.$$

The proof of this proposition is similar to that of the corresponding proposition for a succession of *numbers*

$$a_1, a_2, \cdots,$$

and may be omitted here.

2. DEFINITION. *A set of functions*

$$u_1(x), \ u_2(x), \cdots, \ u_s(x), \cdots,$$

all lying between two fixed limits l and L, is said to be *equably continuous* if

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* §VII is a translation of certain parts of the author's article "Sulle funzioni di linee," *Bologna, Accademia delle Scienze, Memorie*, ser. 5, vol. 5 (1894), p. 225.

A more general proposition than 3 is demonstrated in the memoir "Sulle esistenza degli integrali nelle equazioni differenziali ordinarie," *Ibid*, vol. 6 (1896), p. 131; also in the memoir "Sulle serie di funzioni," *Ibid*, vol. 8(1899), p. 131 and p. 701; and (for the enunciation at least) in the note "Sul secondo teorema della media per gli integrali doppi," *Ibid*, vol. 10 (1902), p. 99.

Concerning propositions 3 and 5 and also Osgood's theorem it may be noted that an analogous but somewhat less general theorem was stated by de la Vallée-Poussin in 1893.

The consideration of the derivatives for the purpose of determining the equable continuity of an infinite set of functions occurs, though under the restrictive hypothesis of the monotonic increase or decrease of all the functions, in the memoir of Ascoli, "Le curve limite di una varietà data di curve," *Atti della R. Accad. dei Lincei, Memorie*, ser. 3, vol. 18 (1882-83), p. 521; see also *R. Istituto Lombardo, Rendiconti*, ser. 2, vol. 21 (1884).

for every arbitrarily chosen positive number, σ, there always exists a positive number, δ, such that, in every interval of length less than δ, all the functions $u_s(x)$ oscillate by less than σ.

3. THEOREM. The necessary and sufficient condition that a set of continuous functions admit a continuous limiting function is that there exist among the set an infinite number of functions which are equably continuous.

The condition is *necessary*.

For suppose $v(x)$ be a limiting function for the functions

$$(a) \qquad u_1(x), \quad u_2(x), \cdots$$

of the succession.

Taking σ arbitrarily we can by (1) find the corresponding m_σ. Since $u_{m_\sigma}(x)$ is continuous, we can find a δ such that, in every interval of length less than δ, $u_{m_\sigma}(x)$ will oscillate by less than σ.

It follows from the inequality in (1) that each of the functions

$$u_{m_\sigma+1}(x), \quad u_{m_\sigma+2}(x), \cdots, \quad u_{m_\sigma+p}(x), \cdots$$

oscillates by less than 3σ in every interval of length less than this same δ.

The functions

$$u_1(x), \quad u_2(x), \cdots, \quad u_{m_\sigma-1}(x)$$

preceding $u_{m_\sigma}(x)$ are finite in number, and hence we can find a number δ_0 such that in every interval of length less than δ_0 they oscillate by less than 3σ. Hence in every interval of length less than the smaller of the two numbers δ and δ_0 each of the functions

$$u_1(x), \quad u_2(x), \cdots, \quad u_n(x), \cdots$$

oscillates by less than 3σ. But 3σ may be *any number whatever*, and, consequently, the equable continuity of the functions is established.

The condition is therefore *necessary*. It is also *sufficient*.

For suppose the functions

$$u_1(x), \quad u_2(x), \cdots, \quad u_n(x), \cdots$$

are equably continuous. Then, given the arbitrarily small positive number σ, we can find a positive number δ such that each of the functions $u_n(x)$ can oscillate by more than σ only in intervals of length greater than δ. For since the functions $u_n(x)$ are equably continuous, we can find for every positive σ a δ such that $u_n(x)$ oscillates by less than σ in every interval of length less than δ.

Consider now the absolute value,

$$|u_p(x) - u_q(x)|,$$

of the difference between any two functions of the set. This difference may in some point x be less than σ, in others equal to or greater than σ.

Let us choose now from the given functions a all those functions,

$$(\beta) \qquad u_{s_1}(x), \quad u_{s_2}(x), \cdots,$$

such that the difference,

$$|u_{s_p}(x) - u_{s_q}(x)|,$$

of any two of them is in some point x greater than or equal to σ, but in all other points is less than σ. Then we can show that the succession β contains at most a finite number of functions.

The functions β are by hypothesis equably continuous. Each of them can therefore oscillate by more than $\sigma/4$ only if x traverses an interval of length greater than a certain assignable value δ'.

Hence if x is a point in which

$$|u_{s_p}(x) - u_{s_q}(x)| \geqq \sigma$$

there will be an interval at least equal to δ' in which the two functions differ by a quantity not less than $\sigma/2$.

Suppose now the functions β were infinite in number. The differences

$$|u_{s_1}(x) - u_{s_2}(x)|, \ |u_{s_1}(x) - u_{s_3}(x)|, \cdots$$

are each, in some interval δ', always greater than or equal to $\sigma/2$; consequently there is a point x_0 which belongs to an infinite number of these intervals.* Therefore, in this point x_0, $u_{s_1}(x)$ differs from an infinite number of the functions β by more than $\sigma/2$ and consequently in a neighborhood of length δ'' about the point x_0 it differs from them by more than $\sigma/4$.

Let the functions from which $u_{s_1}(x)$ thus differs be

$$(\beta') \qquad u_{t1}, \quad u_{t_2}, \cdots.$$

Of the differences,

$$u_{s_1} - u_{t_1}, \qquad u_{s_1} - u_{t_2}, \cdots$$

*This obvious fact may be regarded as a special case of a more general theorem. See: *Rendiconti della R. Accad. dei Lincei, ser. 4, vol. 1, p. 262, 1885, or Memorie della R. Accad. delle Scienze. Bologna, ser. 5, vol. 8, p. 130, 1899.*

there will be an infinite number having the same sign, *i. e.*, there will certainly be either an infinite number of functions greater than u_{s_1}, or an infinite number less than u_{s_1}.

Suppose, for example, there are an infinite number greater than u_{s_1}, and denote them by

$$(\beta'') \qquad u_{r_1}, u_{r_2}, \cdots.$$

Starting from u_{r_1} as previously from u_{s_1} we may reason on the β'' as we have already done on the β; and we find that among the β'' there are an infinite number which in an interval δ'' differ from u_{r_1} by more than $\sigma/4$, the differences being all of the same sign. These functions we may denote by

$$(\beta''') \qquad u_{i_1}, u_{i_2}, \cdots.$$

The reasoning may be continued indefinitely in the same way.

But now we notice that between the curve representing u_{s_1} and the set of curves representing functions of the set β'' there lies an area of magnitude $\omega_1 \geqq \delta'' \sigma/4$ bounded by two curves parallel to u_{s_1} and by two ordinates.

Between u_{r_1}, which is one of the β'', and the set β''' there is similarly an area

$$\omega_2 \geqq \delta'' \frac{\sigma}{4};$$

and, moreover, ω_2 is external to ω_1.

This reasoning we may continue indefinitely, obtaining a set of mutually exclusive areas $\omega_1, \omega_2 \cdots$ each greater than $\delta''\sigma/4$. But these areas must lie in a region of finite area. Therefore they cannot be infinite in number. Hence the functions β are finite in number.

There may be several or an infinite number of sets of functions, which, like the functions β, have the property of differing, each from each of the others by more than σ in some point, but the number of functions composing any one set cannot exceed

$$\frac{4(L-l)\,(b-a)}{\delta''\sigma}.$$

Let one of these sets be

$$(\beta) \qquad u_{s_1}(x), \quad u_{s_2}(x), \cdots, \quad u_{s_p}(x).$$

Of the remaining functions there can be none which differ from these by more than σ in any point, for if there were we should annex them to the set.

It follows that if we consider the neighborhoods

$$u_{s_1}(x) - \sigma \cdots u_{s_1}(x) + \sigma,$$
$$u_{s_2}(x) - \sigma \cdots u_{s_2}(x) + \sigma,$$
$$\cdots\cdots\cdots\cdots$$
$$u^s_p(x) - \sigma \cdots u_{s_p}(x) + \sigma,$$

then in one at least of these neighborhoods there must fall an infinite number of functions belonging to the given set. For if this were not true there would be an infinite number of functions differing from each of the β by more than σ in some point, which is impossible.

We have therefore proved that in the given set x there is a set of functions

$$(a') \qquad u_{m_1}(x), \quad u_{m_2}(x), \cdots,$$

infinite in number, which in every point x differ from each other by less than 2σ. Similarly we may prove that in the set a' there is a sub-set a'' of functions which, taken in pairs, differ in every point x by less than σ^2. In the set a'' we can then find a sub-set a''', the functions in which differ from one another by less than σ^3 in every point. This reasoning may be indefinitely continued.

Let us take now a function $u_{t_1}(x)$ from the set a (for example, u_{t_1} may be the function of lowest index), a function $u_{t_2}(x)$ from the set a', a function $u_{t_3}(x)$ from the set a'', etc. Then if $\sigma < 1$ the set

$$u_{t_1}(x), \quad u_{t_2}(x), \quad u_{t_3}(x), \cdots$$

satisfies the conditions of §VII, and consequently converges to a continuous limiting function, namely, to the function formed by the series

$$u_{t_1}(x) + \Big((u_{t_2}(x) - u_{t_1}(x)\Big) + \Big(u_{t_3}(x) - u_{t_2}(x)\Big) \cdots .$$

The condition is therefore *sufficient* and the theorem is proved.

It is obvious that there may be several or even an infinite number of limiting functions to which converge sets of functions taken from the set a.

4. Consider now, more generally, a set of functions defined by a certain law in the interval $a \cdots b$, all of them being contained between finite limits l and L. Let $u(x)$ be any one of them and denote the set by

$$G \equiv \{u(x)\}.$$

Two functions, $\phi(x)$ and $\psi(x)$ are said to determine a neighborhood of the function $v(x)$ if $\phi(x) < v(x) < \psi(x)$.

If now, however $\phi(x)$ and $\psi(x)$ be taken, provided merely that they satisfy the above inequality, there exist an infinite number of functions of the set G such that $\phi(x) < u(x) < \psi(x)$, then $v(x)$ is said to be a limiting function of the set. It need not however be, itself, a function of the set.

We see at once that, if G contains an equably continuous set of functions, it admits at least one limiting function.

5. THEOREM.* An infinite set of continuous functions will be equably continuous in the interval $a \cdots b$, if the difference quiotient

$$\frac{u(x_1) - u(x_2)}{x_1 - x_2}$$

is, for every $u(x)$ of the set, and for every pair of values x_1 and x_2 possible in the interval $a \cdots b$, comprised between two finite numbers l and L.

For let $|L|$ be the greater of the two numbers $|l|$ and $|L|$. Let h be an arbitrarily taken interval lying in the interval $a \cdots b$; and x_1 and x_2, those points of h which give $u(x)$ its maximum and minimum values respectively.

Then

$$l \leqq \frac{u(x_1) - u(x_2)}{x_1 - x_2} \leqq L.$$

Whence

$$\frac{u(x_1) - u(x_2)}{|x_1 - x_2|} \leqq |L|.$$

Therefore

$$u(x_1) - u(x_2) \leqq |x_1 - x_2| \ |L|,$$

and

$$D_{u,h} \leqq h|L|,$$

$D_{u,h}$ being the oscillation of $u(x)$ in the interval h.

Since this is true from every function $u(x)$ of the set, it follows that, σ being taken arbitrarily small, then in every interval of length

$$h < \frac{\sigma}{|L|}$$

$u(x)$ will oscillate by less than σ. But this is the condition for equable continuity.

BOLOGNA, ITALY,
DECEMBER, 1902.

*Arzelà, *loc. cit.*, p. 232.

GRAPHS OF THE FUNCTIONS Π AND Ψ

By E. P. R. Duval

The diagram on the opposite page consists of two graphs drawn accurately to scale. The curves printed in full lines represent (for values of x between $+3$ and -6) Gauss's Π-function, defined as an infinite product by the formula

$$\Pi(x) = \frac{1}{x+1} \cdot \prod_{n=2}^{n=\infty} \frac{n^{x+1}}{(n-1)^x (x+n)} = \Gamma(x+1).$$

The dotted curves represent (within the same limits) the function

$$\Psi(x) = \frac{d}{dx}\Big(\log \Pi(x)\Big).$$

The graphs were plotted by means of Gauss's tables for $\Pi(x)$ and $\Psi(x)$ for values of x between 0 and 1, and the relations

$$\Pi(x+1) = (x+1)\ \Pi(x),$$

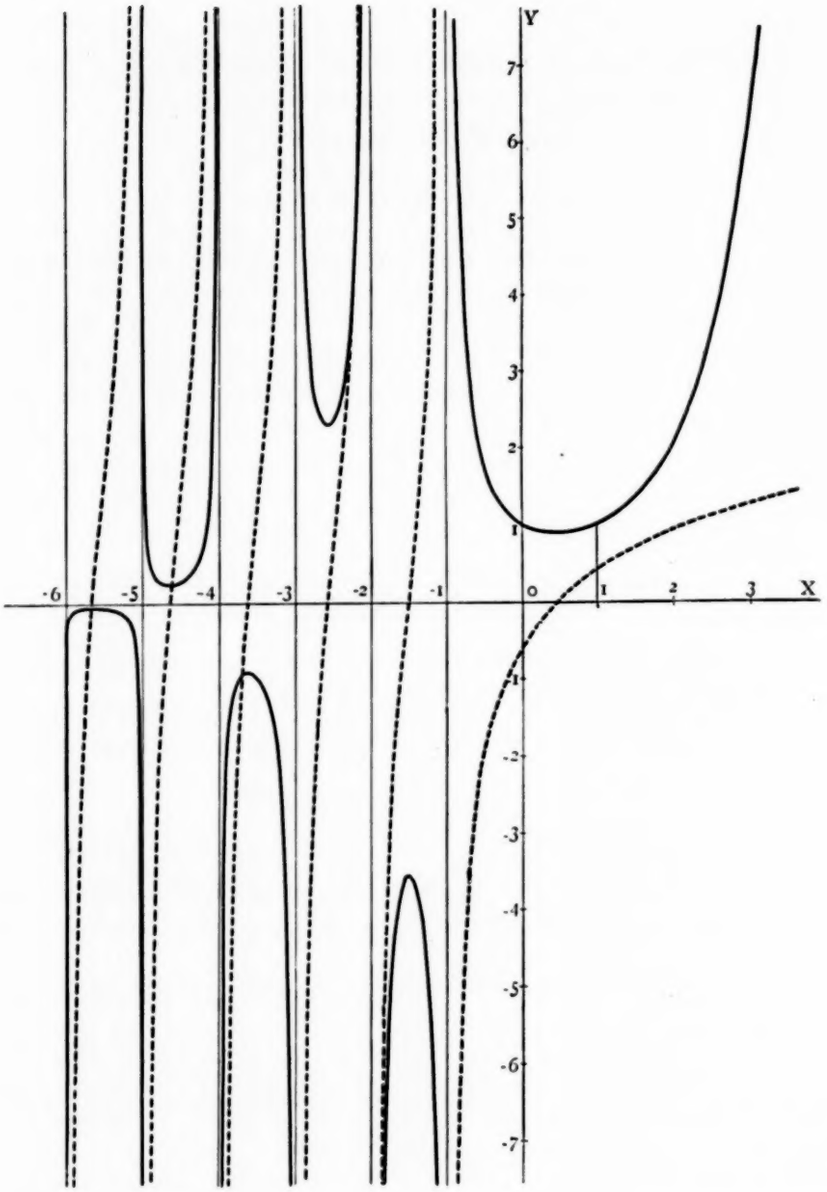
$$\Psi(x+1) = \frac{1}{x+1} + \Psi(x).$$

The unit of length is the distance between two successive vertical asymptotes.

$\Pi(x)$ evidently has maximum or minimum points for all values of x for which $\Psi(x) = 0$. It was partly to illustrate this fact that the two graphs were drawn on the same pair of axes.

The only similar figure which the writer has seen is a somewhat rough graph of $\Gamma(x)$ in Maurice Godefroy's "La Fonction Gamma," page 63, or his "Théorie Elementaire des Séries," page 250.

Harvard University,
Cambridge, Massachusetts.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EXAMPLES OF NON-APPLICABLE SURFACES HAVING THE SAME GAUSSIAN CURVATURE AT CORRESPONDING POINTS

By Arthur Sullivan Gale

Stäckel and Wangerin* have given examples of pairs of non-applicable surfaces which have the same total curvature at corresponding points; *i. e.*, examples showing that the necessary condition for applicability afforded by Gauss's theorem is not sufficient in general, although it is sufficient for surfaces of constant curvature.† An exceedingly simple example of such surfaces is furnished by the following *system* of paraboloids,

$$x = e^{a}u$$
$$y = e^{-a}v$$
$$z = ae^{a}u^{2} + be^{-a}v^{2},$$

where a is a real parameter. That these surfaces are not applicable follows at once from the fact that the square of the element of arc,

$$ds^2 = e^{2a}(1 + 4a^2u^2)du^2 + 8abuvdudv + e^{-2a}(1 + 4b^2v^2)dv^2,$$

is not independent of a. The Gaussian curvature is, however, the same at corresponding points of all the surfaces of the system since it is given by

$$K = \frac{4ab}{(1 + 4a^2u^2 + 4b^2v^2)^2},$$

an expression independent of a.

In a recent note‡ a far more general system of surfaces possessing these properties was considered but attention was not explicitly directed to this point.

Yale University,
September, 1903.

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* Two notes, Zur Theorie des Gauss'chen Krümmungsmaasses, *Leipziger Berichte*, vol. 45 (1893), pp. 163 and 170 respectively. See also Scheffers, *Anwendung der Differential- und Integralrechnung auf Geometrie*, vol. 2, p. 276.

† Bianchi, *Vorlesungen über Differentialgeometrie*, p. 186.

‡ Annals of Mathematics, vol. 4(1902-03), p. 107; in particular, see the theorems on p. 113.

THE MATHEMATICAL THEORY OF THE TOP

By A. G. Greenhill

(*Continued from page 20*)

17. The dynamical constants, G, G', E, being three in number, there is a triply infinite (∞^3) assemblage of states of motion of the top, so that the selection of an illustrative numerical example is at first sight an embarrassing one in its variety.

But the student requires some well selected numerical examples of an actual case of motion, to fix his ideas and to refer to for elucidation, and this can be obtained by choosing for the parameter v in (13) §9 a simple aliquot part of a period, so that f is a rational fraction; in this way we can utilize Abel's theory, and replace the theta functions in the general integral $I(v)$ of (1) and (2) §10 by algebraical functions.

Beginning with the simplest case, take

$$GE - G' = 0, \quad Q = 0; \tag{1}$$

then v is a half-period, either ω_1 or ω_2 but never ω_3; $f = 0$ or 1;
$E = z_1$ or z_2, never z_3 for real motion, because $E - z$ must be positive;

$$x_1 = 0 \quad \text{or} \quad x_2 = 0, \qquad \rho_1 = 0 \quad \text{or} \quad \rho_2 = 0, \tag{2}$$

$$\rho = \rho_3 \mathrm{dn}(K - mt), \quad \text{or} \quad \rho_3 \mathrm{cn}(K - mt), \tag{3}$$

and replacing the k of § 7 by the letter c, to avoid confusion with the modulus of the elliptic functions,

$$\omega = \frac{\delta}{c} nt = \frac{L}{M} nt = pt, \tag{4}$$

suppose, giving the curve of H; and in the associated motion of the top

$$\chi = pt - \psi. \tag{5}$$

$$\tfrac{1}{2} M^2 \sin\theta \, e^{\chi i} = L\sqrt{(x - x_1)} + i\sqrt{(x_3 - x)(x - x_2)}, \tag{6}$$

$$\text{or} \quad L\sqrt{(x - x_2)} + i\sqrt{(x_3 - x)(x - x_1)};$$

$$\sin\theta \sin(pt - \psi) = \sqrt{(z_2 - z)(z - z_3)} \quad \text{or} \quad \sqrt{(z_1 - z)(z - z_3)}, \tag{7}$$

$$(8)\qquad \sin\theta\cos(pt-\psi)=\frac{L}{M}\sqrt{2(z_1-z)},\qquad \text{or}\qquad \frac{L}{M}\sqrt{2(z_2-z)},$$

$$(9)\qquad \frac{p}{n}=\frac{L}{M}=\sqrt{\frac{z_2+z_3}{2}},\qquad \text{or}\qquad \sqrt{\frac{z_1+z_3}{2}},$$

$$(10)\qquad \frac{G'}{G}=z_1=\frac{1+z_2z_3}{z_2+z_3},\qquad \text{or}\qquad \frac{G'}{G}=z_2=\frac{1+z_1z_3}{z_1+z_3}.$$

18. Next take the bisection of a half-period where $f=\frac{1}{2}$:

$$(1)\qquad r=\omega_1+\tfrac{1}{2}\omega_3,\qquad \text{or}\qquad K+\tfrac{1}{2}K'i,$$

and work with

$$(2)\qquad I=\tfrac{1}{2}\int_x^k\frac{(1-k)(k+x^2)}{k-x^2}\,\frac{dx}{\sqrt{X}},$$

$$(3)\qquad X=(1-x^2)(k^2-x^2),\quad P=\tfrac{1}{2}(1-k),\quad Q=k(1-k),$$

$$(4)\qquad pt-\omega=I=\cos^{-1}\sqrt{\frac{(1-x)(k+x)}{2(k-x^2)}}=\sin^{-1}\sqrt{\frac{(1+x)(k-x)}{2(k-x^2)}},$$

$$(5)\qquad nt=\int_x^k\frac{M\,dx}{\sqrt{X}},\qquad \frac{p}{n}=\frac{L-\frac{1}{2}(1-k)}{M},$$

$$(6)\qquad x=k\,\mathrm{sn}(K-mt),\quad n=Mm;$$

and the curve of H is given by

$$(7)\qquad \frac{\rho}{c}e^{(pt-\omega)i}=\frac{\sqrt{\frac{1}{2}(1-x)(k+x)}+i\sqrt{\frac{1}{2}(1+x)(k-x)}}{M},$$

$$(8)\qquad \frac{\rho^2}{c^2}=\frac{k-x^2}{M^2},$$

$$(9)\qquad e^{(pt-\omega)i}=\frac{\sqrt{\frac{1}{2}(1-x)(k+x)}+i\sqrt{\frac{1}{2}(1+x)(k-x)}}{\sqrt{(k-x^2)}},$$

$$(10)\qquad \sin 2(pt-\omega)=\frac{\mathrm{cn}(K-mt)\,\mathrm{dn}(K-mt)}{1-k\,\mathrm{sn}^2(K-mt)}$$

$$=\frac{(1+k)\,\mathrm{sn}\,mt}{1+k\,\mathrm{sn}^2\,mt}$$

$$=\mathrm{sn}[(1+k)mt,\gamma],\qquad \gamma=\frac{2\sqrt{k}}{1+k},$$

by Landen's Second Transformation:

$$(11)\qquad pt-\omega=\tfrac{1}{2}\mathrm{am}[(1+k)nt,\gamma].$$

In the associated motion of the top, derived in the preceding manner,

$$(12)\quad \tfrac{1}{2}M^2\sin\theta\, e^{(pt-\psi)i} = (L-1+k-x)\sqrt{\tfrac{1}{2}(1-x)(k+x)} + i(L-1+k+x)\sqrt{(\tfrac{1}{2}(1+x)(k-x)},$$

$$(13)\quad M^2\cos\theta = L^2 - 1 + k - k^2 + 2x^2,$$

$$(14)\quad (\tfrac{1}{2}M^2\sin\theta)^2 = k(L-1+k)^2 - (L^2-1+k-k^2)x^2 - x^4,$$

$$(15)\quad M^2 z_1 = L^2 + 1 + k - k^2, \quad \tfrac{1}{2}M^2\mathrm{sh}\,\theta_1 = (L+k)\sqrt{(1-k)},$$

$$(16)\quad M^2 z_2 = L^2 - 1 + k + k^2, \quad \tfrac{1}{2}M^2\sin\theta_2 = (L-1)\sqrt{(k-k^2)},$$

$$(17)\quad M^2 z_3 = L^2 - 1 + k - k^2, \quad \tfrac{1}{2}M^2\sin\theta_3 = (L-1+k)\sqrt{k};$$

$$(18)\quad M^2 E = L^2 - 1 + 3k - k^2,$$

$$(19)\quad M^4 = (L^2 - 1 + 3k - k^2)^2 - 8k(1-k)L + 8k(1-k)^2.$$

$$(20)\quad L' = \frac{L^3 - (1-3k+k^2)L - 2k(1-k)}{M^2}.$$

Thus $L' = L = -k$ gives an intermediate path, and we find

$$(21)\quad pt - \psi = \tfrac{3}{4}\pi + \tfrac{1}{2}\,\mathrm{am}\,[\Gamma - (1+k)\,mt,\ \gamma];$$

$L' = L = 1$ gives an upper rosette, with

$$(22)\quad pt - \psi = -\tfrac{1}{4}\pi + \tfrac{1}{2}\,\mathrm{am}\,[\Gamma - (1+k)\,mt,\ \gamma],$$

and $L' = -L = -1 + k$ a lower rosette, with

$$(23)\quad pt - \psi = \pi - \tfrac{1}{2}\,\mathrm{am}\,[(1+k)\,mt,\ \gamma];$$

these rosette curves are shown in a stereoscopic diagram in *Engineering*, 10 Oct., 1897, for $k = 0.77384$, in the upper rosette, and $k = 0.42893$ in the lower, thus making the apsidal angle $\frac{4}{5}\pi$ and $\frac{2}{5}\pi$.

Cusps occur on the circle z_2 for

$$(24)\quad L = 1 \pm \sqrt{(k+k^2)}.$$

The axis moves like a spherical pendulum when $L' = 0$; and putting

$$(25)\quad L = c(1-k),$$

$$(26)\quad \frac{1}{k} + k = \frac{2c^3 - 3c + 2}{c^3 - c}, \quad \frac{2\sqrt{k}}{1+k} = 2\sqrt{\frac{c(c^2-1)}{(2c-1)(2c^2+c-2)}},$$

in which c may be taken as the arbitrary constant for the construction of numerical cases. (*Proc. Lond. Math. Soc.*,* vol. 27, p. 610.)

The motion is purely algebraical when

$$p = 0, \qquad L = \tfrac{1}{2}(1 - k), \tag{27}$$

$$(2M)^4 = (1 + k)^2(9 - 14k + 9k^2), \tag{28}$$

$$(2M)^2 z_1 = (1 + k)(5 - 3k), \tag{29}$$

$$(2M)^2 z_2 = (1 + k)(-3 + 5k), \tag{30}$$

$$(2M)^2 z_3 = -3 + 2k - 3k^2, \tag{31}$$

$$(2M)^2 E = -3 + 10k - 3k^2, \tag{32}$$

$$L' = -\frac{3(1 + k)^2(1 - k)}{8M^2}. \tag{33}$$

Stereoscopic figures of this case, drawn by Mr. T. I. Dewar, are given in L. M. S., vol. 27, p. 587, drawn for

$$k = \frac{15}{17}, \quad \frac{3}{5}, \quad \text{and} \quad \frac{1}{3}\ (\text{cusps}). \tag{34}$$

We take $k = \frac{3}{5}$ as affording a good illustrative diagram of the general case: then

$$L = \frac{1}{3}, \quad L' = -\frac{\sqrt{6}}{5}, \quad M = \frac{2\sqrt{24}}{5}, \quad D = \frac{\sqrt{6}}{24}, \quad E = \frac{\sqrt{6}}{4}, \quad F = \frac{7\sqrt{6}}{24}, \tag{35}$$

$$z_1 = \frac{2\sqrt{6}}{3}, \quad z_2 = 0, \ \theta_2 = 90^\circ, \quad z_3 = -\frac{3}{8}\sqrt{6} = \cos 156^\circ 44'; \ \sin\theta_3 = \frac{\sqrt{10}}{8}, \tag{36}$$

and the curve of H and projection P of C' are shown in figure 5 (at the end of the article), and can be realized experimentally with the bicycle wheel, suspended from a bracket, as shown in figures 3 and 4 (page 19).

In this case the points of half time and of inflexion coincide at h, but this is accidental: the representative points H, K, P are taken for

$$x = \frac{1}{10}; \text{ and then } mt = \frac{9}{10}K \text{ approximately.}$$

19. For the trisection of a period, where

$$v = \omega_1 + f\omega_3, \quad f = \tfrac{1}{3} \text{ or } \tfrac{2}{3}, \tag{1}$$

* Abbreviated to L. M. S.

we can utilize the forms given in L. M. S., vol. 25, p. 218, for $\mu = 6$; and changing c there into $\frac{b-1}{b+1}$, put

$$P = \frac{1}{6}(9-b^2),\ Q = \tfrac{1}{2}(9-b^2)(b^2-1), \tag{2}$$

$$x_2 = \tfrac{1}{4}(9-b^2)(b^2-1),\ x_3 = (3+b)(b-1),\ x_1 = -(3-b)(b+1), \tag{3}$$

$$I = \int_{x_2}^{x} \frac{-\frac{1}{6}(9-b^2)x + \frac{1}{2}(9-b^2)(b^2-1)}{x} \frac{dx}{\sqrt{X}} \tag{4}$$

$$= \tfrac{1}{3}\cos^{-1}\frac{\frac{1}{2}(9-b^2)\sqrt{(x_3-x)(x-x_1)}}{x^2}$$

$$= \tfrac{1}{3}\sin^{-1}\frac{(x+9-b^2)\sqrt{(x-x_2)}}{x^2};$$

so that the curve of H is given by

$$M\frac{p}{c}e^{(pt-\omega)i} = \left\{\tfrac{1}{2}(9-b^2)\sqrt{(x_3-x)(x-x_1)} + i(x+9-b^2)\sqrt{(x-x_2)}\right\}^{\frac{1}{3}}, \tag{5}$$

$$\frac{p}{n} = \frac{L - \frac{1}{6}(9-b^2)}{M}. \tag{6}$$

The associated motion of the top is given by equation (3) §13,

$$\tfrac{1}{2}M^2\sin\theta\, e^{(pt-\psi)i} = \frac{Lx - \frac{1}{2}(9-b^2)(b^2-1) + \frac{1}{2}i\sqrt{X}}{\sqrt{x}} \times\left\{\frac{\frac{1}{2}(9-b^2)\sqrt{(x_3-x)(x-x_1)} + i(x+9-b^2)\sqrt{(x-x_2)}}{x^2}\right\}^{\frac{1}{3}}, \tag{7}$$

$$M^2\cos\theta = L^2 - \tfrac{1}{4}(9-b^2)^2 + 12 - 2x, \tag{8}$$

$$M^4 = \left\{L^2 - \tfrac{1}{4}(9-b^2)^2 + 12\right\}^2 - 4(9-b^2)(b^2-1)L - 2(9-b^2)(b^2-1)(b^2-5), \tag{9}$$

and working this out we find

$$(\tfrac{1}{2}M^2\sin\theta)^3 e^{3(pt-\psi)i} = R\sqrt{(x_3-x)(x-x_1)} + iS\sqrt{(x-x_2)}, \tag{10}$$

$$(\tfrac{1}{2}M^2\sin\theta)^2 = -x^2 + \left\{L^2 - \tfrac{1}{4}(9-b^2)^2 + 12\right\}x - (9-b^2)(b^2-1)L - \tfrac{1}{2}(9-b^2)(b^2-1)(b^2-5), \tag{11}$$

$$R = -x^2 - \left\{3L^2 - \tfrac{3}{2}(9-b^2)L + \tfrac{1}{2}(b^2-3)(b^2-13)\right\}x + \tfrac{1}{2}(9-b^2)\left\{[L + \tfrac{1}{2}(b^2-5)]^3 - (b^2-1)^2\right\}. \tag{12}$$

$$(13)\quad S = [3L - \tfrac{1}{2}(9-b^2)]x^2 + \{L^3 - \tfrac{3}{2}(9-b^2)L^2 + \tfrac{3}{4}(b^4 - 22b^2 + 69)L + \tfrac{1}{8}(9-b^2)^2(b^2-5)\}\{x + (9-b^2)\}\{[L + \tfrac{1}{2}(b^2-5)]^3 + (b^2-1)^2\},$$

a factor x^3 cancelling.

Arranged in powers of L.

$$(14)\quad R = \tfrac{1}{2}(9-b^2)L^3 - 3\{x - \tfrac{1}{4}(9-b^2)(b^2-5)\}L^2 + \tfrac{3}{2}(9-b^2)\{x + \tfrac{1}{4}(b^2-5)^2\}L - x^2 - \tfrac{1}{2}(b^2-3)(b^2-13)x + \tfrac{1}{8}(9-b^2)\{\tfrac{1}{8}(b^2-5)^3 - (b^2-1)^2\},$$

$$(15)\quad S = (x + 9 - b^2)L^3 - \tfrac{3}{2}(9-b^2)(x - b^2 + 5)L^2 + 3\{x^2 + \tfrac{1}{4}(b^4 - 22b^2 + 69)x + \tfrac{1}{4}(9-b^2)(b^2-5)^2\}L + (9-b^2)\{-\tfrac{1}{2}x^2 + \tfrac{1}{8}(9-b^2)(b^2-5)x + \tfrac{1}{8}(b^2-5)^3 + (b^2-1)^2\}.$$

In the upper rosette

$$(16)\quad L = \frac{Q}{x_2} = 2, \quad M = \tfrac{1}{2}(b^2-1),$$

$$(17)\quad (M\cos\tfrac{1}{2}\theta)^3 e^{3i(pt-\psi)i} = \tfrac{1}{2}\sqrt{X} + i\{\tfrac{1}{2}(b^2+3)x - \tfrac{1}{2}(b^2-1)(b^2+7)\},$$

$$(18)\quad \frac{p}{n} = \frac{b^2+3}{6M} = \frac{b^2+3}{3(b^2-1)}.$$

In the lower rosette

$$(19)\quad L = \frac{Q}{x_3} = \tfrac{1}{2}(3-b)(b+1), \quad M^2 = b(b-1)^2,$$

$$(20)\quad (M\sin\tfrac{1}{2}\theta)^3 e^{3i(pt-\psi)i} = [x - \tfrac{1}{2}(3-b)(b-1)(b^2+b+2)]\sqrt{(x-x_1)} + i(3b-b^2)\sqrt{(x_3-x)(x-x_2)},$$

$$(21)\quad \frac{p}{n} = \frac{3b-b^2}{3M}.$$

In the intermediate orbit

$$(22)\quad L = \frac{Q}{x_1} = -\tfrac{1}{3}(3+b)(b-1), \quad M^2 = b(b+1)^2,$$

$$(23)\quad (M\cos\tfrac{1}{2}\theta)^3 e^{3i(pt-\psi)i} = [-x - \tfrac{1}{2}(3+b)(b+1)(b^2-b+2)]\sqrt{(x_3-x)} - i(3b+b^2)\sqrt{(x-x_2)(x-x_1)},$$

$$(24)\quad \frac{p}{n} = \frac{-3b-b^2}{3M}.$$

The axis moves like a spherical pendulum when

$$L' = 0, \quad L^3 - \tfrac{1}{4}(9 - b^2)^2 L + 12L - (9 - b^2)(b^2 - 1) = 0, \tag{25}$$

a quadratic for b^2 in terms of L, so that L may be taken as the parameter.

The conditions (3) and (8) §16 for cusps on $z = z_2$ and $z = z_1$ become

$$(L - 2)^2 = \tfrac{1}{4}(b^2 - 1)^2, \quad L = \frac{5 - b^2}{2}, \ \frac{3 + b^2}{2}; \tag{26}$$

$$[L + \tfrac{1}{2}(3 + b)(b - 1)]^2 = b(b + 1)^2. \tag{27}$$

The secular term pt is cancelled by putting

$$L = \tfrac{1}{6}(9 - b^2); \tag{28}$$

specimen cases of this nature in stereoscopic representation are given in L. M. S., vol. 27, pp. 592, 593, having six branches for the region $3 > b > 1$, and three branches for the region $\infty > b > 3$.

In the algebraical case of six cusps,

$$b^2 = 3, \quad L = 1, \quad M^4 = 16, \quad L' = -2, \tag{29}$$

$$x_3 = 2\sqrt{3}, \quad x_2 = 3, \quad x_1 = -2\sqrt{3}, \quad A = 3, \quad B = -12, \tag{30}$$

$$z_1 = 1 + \sqrt{3}, \quad z_2 = -\tfrac{1}{2} = \cos 120°, \quad z_3 = 1 - \sqrt{3} = \cos 137°, \tag{31}$$

$$\cos\theta = 1 - \tfrac{1}{2}x, \quad \sin^2\theta = x - \tfrac{1}{4}x^2, \tag{32}$$

$$\sin^3\theta\, e^{3\psi i} = \tfrac{1}{8}(-x^2 + 6x - 12)\sqrt{(-x^2 + 12)} + i\,(x - 3)^{\frac{3}{2}}. \tag{33}$$

With the other value for L in (26) for cusps

$$b^2 = 3, \quad L = 3, \quad M^4 = 48, \quad L' = 2\sqrt{3}, \tag{34}$$

$$z_1 = \sqrt{3} + 1, \quad z_2 = \tfrac{1}{2}\sqrt{3} = \cos 30°, \quad z_3 = \sqrt{3} - 1 = \cos 43°, \tag{35}$$

$$\cos\theta = \sqrt{3} - \frac{x}{2\sqrt{3}}, \quad 12\sin^2\theta = -x^2 + 12x - 24, \tag{36}$$

$$(2\sqrt{3}\sin\theta)^3 e^{3\left(\frac{mt}{\sqrt{3}} - \psi\right)i} = (-x^2 + 12)^{\frac{3}{2}} + 6i(x^2 - 6x + 12)\sqrt{(x - 3)}, \tag{37}$$

an illustrative numerical case of quasi-algebraical motion, with cusps at angular interval of about 79°.

For the algebraical case of three cusps we obtain from (27)

$$L + \tfrac{1}{2}(3 + b)(b - 1) = \tfrac{1}{3}(b^2 + 3b) = (b + 1)b^{\frac{1}{2}}, \tag{38}$$

$$(b^{\frac{1}{2}} - 1)^3 - 2 = 0, \quad b = (\sqrt[3]{2} + 1)^2, \quad L = -\sqrt[3]{4} - \sqrt[3]{2}. \tag{39}$$

In the region $3 > b > 1$,

$$x_3 > x > x_2 > 0 > x_1, \tag{40}$$

$$k^2 = \frac{x_3 - x_2}{x_3 - x_1} = \frac{(3+b)(b-1)^3}{16b}, \quad k'^2 = \frac{(3-b)(b+1)^3}{16b}, \tag{41}$$

$$\operatorname{sn}\tfrac{1}{3}K' = \frac{2}{b+1}, \quad \operatorname{cn}\tfrac{2}{3}K' = \frac{b-1}{b+1}, \tag{42}$$

$$\operatorname{sn}\tfrac{1}{3}K' + \operatorname{cn}\tfrac{2}{3}K' = 1. \tag{43}$$

20. Generally for a parameter

$$v = \omega_1 + f\omega_3, \quad \text{or} \quad K + fK'i, \quad f = \frac{r}{2n+1}, \tag{1}$$

obtained by the division of a half-period by an odd number $2n+1$, we must proceed to the problem of Division and Transformation of order $\mu = 4n+2$, as explained in L. M. S. vols. 25 and 27, to discover the relations required for

$$I(v) = \int_{x_2}^{x} \frac{-Px + Q}{x} \frac{dx}{\sqrt{X}} \tag{2}$$

$$= \frac{1}{2n+1} \cos^{-1} \frac{P_1 x^{n-1} + P_2 x^{n-2} + \cdots + P_n}{x^{n+\frac{1}{2}}} \sqrt{(x_3 - x)(x - x_1)}$$

$$= \frac{1}{2n+1} \sin^{-1} \frac{x^n + Q_1 x^{n-1} + Q_2 x^{n-2} + \cdots + Q_n}{x^{n+\frac{1}{2}}} \sqrt{(x - x_2)}.$$

A differentiation shows that

$$P_1 = (2n+1)P, \quad \text{and} \quad \frac{P}{\sqrt{(x_3 - x_1)}} = \operatorname{zn} fK', \tag{3}$$

from (4) §10; and for the associated elliptic functions,

$$k^2 = \frac{x_3 - x_2}{x_3 - x_1}, \quad k'^2 = \frac{x_2 - x_1}{x_3 - x_1}; \tag{4}$$

while for the Division-Values to the modulus k',

$$\operatorname{sn}^2 fK' = \frac{-x_1}{x_2 - x_1}, \quad \operatorname{cn}^2 fK' = \frac{x_2}{x_2 - x_1}, \quad \operatorname{dn}^2 fK' = \frac{x_3}{x_3 - x_1}; \tag{5}$$

$$\operatorname{sn}^2(1-f)K' = \frac{x_2}{x_3} \cdot \frac{x_3 - x_1}{x_2 - x_1}, \quad \operatorname{cn}^2(1-f)K' = \frac{-x_1}{x_3} \cdot \frac{x_3 - x_2}{x_2 - x_1}, \tag{6}$$

$$\operatorname{dn}^2(1-f)K' = 1 - \frac{x_2}{x_3}.$$

The curve of H is now given by

$$(7) \qquad M\frac{\rho}{c}\,e^{(pt-\omega)i} = \frac{\theta(0)\;\theta(u-v)}{\lambda\,\theta u\,\theta v}$$

$$= \{E\sqrt{(x_3-x)(x-x_1)} + iF\sqrt{(x-x_2)}\}^{\frac{1}{2n+1}}$$

$$(8) \qquad E = P_1x^{n-1} + P_2x^{n-2} + \cdots + P_n,$$

$$(9) \qquad F = x^n + Q_1x^{n-1} + Q_2x^{n-2} + \cdots + Q_n;$$

and then for the associated motion of the axis of the top

$$(10) \qquad \tfrac{1}{2}M^2\sin\theta\;e^{(pt-\psi)i} = \frac{Lx - Q + \frac{1}{2}i\sqrt{X}}{\sqrt{x}}$$

$$\times\left\{\frac{E\sqrt{(x_3-x)(x-x_1)} + iF(x-x_2)}{x^{n+\frac{1}{2}}}\right\}^{\frac{1}{2n+1}}$$

21. Rationalizing by raising to the power $2n+1$, and putting

$$(1) \qquad (Lx - Q + \tfrac{1}{2}i\sqrt{X})^{2n+1} = H + K\sqrt{T},$$

where

$$(2) \qquad T = -\tfrac{1}{4}X = x^3 - Ax^2 + Bx + Q^2,$$

$$(3) \qquad H = (2n+1)(Lx-Q)T^n$$

$$+ \frac{(2n+1)2n(2n-1)}{3\,!}(Lx-Q)^3T^{n-1}$$

$$+ \frac{(2n+1)2n(2n-1)(2n-2)(2n-3)}{5\,!}(Lx-Q)^5T^{n-2} + \cdots$$

$$= (2n+1)\,Lx^{3n+1} + (2n+1)\left(\frac{2n(2n-1)}{3\,!}L^3 - nAL - Q\right)x^{3n}$$

$$+ H_1x^{3n-1} + \cdots + H_{3n};$$

$$(4) \qquad K = T^n + \frac{(2n+1)2n}{2\,!}(Lx-Q)^2T^{n-1} + \cdots$$

$$= x^{3n} + K_1x^{3n-1} + \cdots + K_{3n}.$$

Then

$$(5) \qquad (\tfrac{1}{2}M^2\sin\theta)^{2n+1}\,e^{(2n+1)(pt-\psi)i} = R\sqrt{(x_3-x)(x-x_1)} + iS\sqrt{(x-x_2)},$$

where

$$R_1x^{2n+1} = EH - F(x-x_2)K = -x^{4n+1} + R_1x^{4n} + \cdots + R_{2n}x^{2n+1} + 0; \tag{6}$$

$$S_1x^{2n+1} = FH - E(x-x_3)(x-x_1)K = S_0x^{4n+1} + S_1x^{4n} + \cdots + S_{2n}x^{2n+1} + 0; \tag{7}$$

the coefficient of powers of x below the $(2n+1)$th cancelling.

The calculation of $R_1, R_2, \cdots R_{2n}$, $S_0, S_1, S_2, \cdots S_{2n}$, is thus a piece of heavy algebraical work, when once E and F have been determined.

A few of the leading coefficients in the general case will serve for verification with the simplest values of the divisor μ; thus

$$S_0 = (2n+1)L - P_1 = (2n+1)(L-P), \tag{8}$$

and

$$pt = (L-P)\int_{x_2}^{x}\frac{dx}{\sqrt{X}}, \tag{9}$$

so that S_0 vanishes in the purely algebraical case where the secular term pt is cancelled by taking

$$L = P. \tag{10}$$

Proceeding with the calculations,

$$R_1 = -(2n+1)nL^2 + (2n+1)P_1L - Q_1 + nA + x_2, \tag{11}$$

$$S_1 = \frac{(2n+1)2n(2n-1)}{3!}L^3 - (2n+1)nP_1L^2 + (2n+1)(Q_1 - nA)L - (2n+1)Q + (nA + x_1 + x_3)P_1 - P_2. \tag{12}$$

$$R_2 = \cdots \quad , \quad S_2 = \cdots . \tag{13}$$

22. Otherwise, arranging H and K in ascending powers of T and descending powers of $Lx - Q$, and writing

$$E(x-x_3)(x-x_1) = E', \quad F(x-x_2) = F' \tag{1}$$

$$Rx^{2n+1} = EH - F'K, \quad Sx^{2n+1} = FH - E'K, \tag{2}$$

then in descending powers of Lx,

$$(3)\qquad Rx^{2n+1} = E(Lx)^{2n+1} - (2n+1)\, r_1 (Lx)^{2n} + \frac{(2n+1)2n}{2!}\, r_2 (Lx)^{2n-1} - \ldots,$$

$$(4)\qquad Sx_{2n}{}^{+1} = F(Lx)^{2n+1} - (2n+1)\, s_1 (Lx)^{2n} + \frac{(2n+1)2n}{2!}\, s_2 (Lx)^{2n-1} - \ldots,$$

$$(5)\qquad r_1 = QE + F', \qquad s_1 = QF + E',$$

$$(6)\qquad r_2 = (T + Q^2)E + 2QF', \qquad s_2 = (T + Q^2)F + 2QE',$$

$$(7)\qquad \begin{aligned} r_3 &= (3QT + Q^3)E + (T + 3Q^2)F', \\ s_3 &= (3QT + Q^3)F + (T + 3Q^2)E', \end{aligned}$$

$$(8)\qquad \begin{aligned} r_4 &= (T^2 + 6Q^2T + Q^4)E + (4QT + 4Q^3)F', \\ s_4 &= (T^2 + 6Q^2T + Q^4)F + (4QT + 4Q^3)E', \end{aligned}$$

$$(9)\qquad \begin{aligned} r_5 &= (5QT^2 + 10Q^3T + Q^5)E + (T^2 + 10Q^2T + 5Q^4)F', \\ s_5 &= (5QT^2 + 10Q^3T + Q^5)F + (T^2 + 10Q^2T + 5Q^4)E', \end{aligned}$$

and so on; and r_1, s_1 have a factor x; r_n, s_n have a factor x^n.

23. Another way of determining the coefficients in R and S is to take

$$(1)\qquad \begin{aligned} pt - \psi &= \frac{1}{2n+1} \cos^{-1} \frac{R\sqrt{(x_3 - x)(x - x_1)}}{\{-x^2 + (L^2 + A)x - 2LQ - B\}^{n+\frac{1}{2}}} \\ &= \frac{1}{2n+1} \sin^{-1} \frac{S\sqrt{(x - x_2)}}{\{-x^2 + (L^2 + A)x - 2LQ - B\}^{n+\frac{1}{2}}}, \end{aligned}$$

$$(2)\qquad \begin{aligned} \frac{d\psi}{dx} &= 2\,\frac{L - L'z}{M \sin^2\theta}\, n\, \frac{dt}{dx} \\ &= \frac{(L^3 + AL - 2Q)x - 3L^2Q - 2LB + AQ}{\frac{1}{4}M^4 \sin^2\theta \sqrt{X}}, \end{aligned}$$

$$(3)\qquad p\frac{dt}{dx} = \frac{L - P}{\sqrt{X}},$$

$$(4)\qquad p\frac{dt}{dx} - \frac{d\psi}{dx} = \frac{-(L - P)x^2 - \{(L^2 + A)P - 2Q\}x + L^2Q + L(B + 2PQ) + BP - AQ}{\{-x^2 + (L^2 + A)x - 2LQ - B\}\sqrt{X}},$$

and thence by differentiation of (1) we can obtain sufficient relations to determine the coefficients of x in R and S, and to complete the verification.

24. These values of R and S are found to verify for the preceding case of $\mu = 6$, worked out independently.

For $\mu = 10$, $f = \frac{1}{5}$, take the expression for $I(c)$ given in L. M. S., vol. 27, p. 601, and put $\sigma - s = x$; then

$$x_2 = (c+1)^3(c-1)(-c^2+4c+1), \tag{1}$$

$$x_3 + x_1 = 8(c^4 + c^3 - 3c^2 - c), \tag{2}$$

$$x_1x_3 = -16c^2(c+1)^3(c-1)(-c^2+4c+1), \tag{3}$$

$$x_3 - x_1 = 16c^2\sqrt{(c^3+c^2-c)}, \tag{4}$$

$$\begin{aligned} A &= -c^6 + 2c^5 + 17c^4 + 12c^3 - 31c^2 - 14c - 1, \\ B &= 8c(c+1)^3(c-1)(-c^2+4c+1)(c^3+c^2-5c-1), \end{aligned} \tag{5}$$

$$P = \tfrac{1}{5}(c+3)(-c^2+4c+1), \tag{6}$$

$$Q = 4c(c+1)^3(c-1)(-c^2+4c+1), \tag{7}$$

$$P_1 = (c+3)(-c^2+4c+1), \tag{8}$$

$$P_2 = 4(c+1)^3(-c^2+4c+1)^2, \tag{9}$$

$$Q_1 = 4(2c^2+5c+1)(-c^2+4c+1), \tag{10}$$

$$Q_2 = 16c(c+1)^3(-c^2+4c+1)^2, \tag{11}$$

$$\begin{aligned} I &= \int_{x_2}^{x} \frac{-Px+Q}{x}\frac{dx}{\sqrt{X}} \\ &= \tfrac{1}{5}\cos^{-1}\frac{P_1x+P_2}{x^{\frac{3}{2}}}\sqrt{(x_3-x)(x-x_1)} \\ &= \tfrac{1}{5}\sin^{-1}\frac{x^2+Q_1x+Q_2}{x^{\frac{3}{2}}}\sqrt{(x-x_2)}. \end{aligned} \tag{12}$$

Then for the curve of H,

$$\left(M\frac{\rho}{k}\right)^5 e^{5i(pt-\omega)} = E\sqrt{(x_3-x)(x-x_1)} + iF\sqrt{(x-x_2)}, \tag{13}$$

$$E = (c+3)(-c^2+4c+1)x + 4(c+1)^3(-c^2+4c+1)^2, \tag{14}$$

$$F = x^2 + 4(-c^2+4c+1)(2c^2+5c+1)x + 16c(c+1)^3(-c^2+4c+1)^2, \tag{15}$$

$$M\frac{\rho}{k} = \sqrt{x}, \quad \frac{p}{n} = \frac{L - \frac{1}{5}(c+3)(-c^2+4c+1)}{M}. \tag{16}$$

For the associated motion of the top,

$$(17)\quad (\tfrac{1}{2}M^2\sin\theta)^5 e^{5(pt-\psi)i} = R\sqrt{(x_3-x)(x-x_1)} + iS\sqrt{(x-x_2)},$$

where, arranged in powers of L,

$$(18)\qquad R = L^5E - 5L^4E_1 + 10L^3E_2 - 10L^2E_3 + 5LE_4 - E_5,$$

$$(19)\qquad S = L^5F - 5L^4F_1 + 10L^3F_2 - 10L^2F_3 + 5LF_4 - F_5,$$

and, employing detached coefficients for the powers of c, for economy of notation,

$$(20)\quad E_1 = x^2 + (-c^2+4c+1)(-c^4-2+8+22+5)x - 4(c+1)^3(-c^2+4+1)^2(c^3+1-5-1),$$

$$(21)\quad F_1 = P_1x^3 + 4(-c^2+4+1)(-2c^5-5+10+32+12+1)x - 16c(c+1)^3(-c^2+4+1)^2(c^3+1-5-1);$$

$$(22)\quad E_2 = P_1x^3 + (-c^2+4+1)(c^3+1-5-1)(c^4+0-14-28-7)x + P_2(c^3+1-5-1)^2,$$

$$(23)\quad F_2 = x^3 + (c^6-2-25+0+115+50+5)x^2 + 4(-c^2+4+1)(c^3+1-5-1)(2c^5+3-16-38-14-1)x + Q_2(c^3+1-5-1)^2;$$

$$(24)\quad E_3 = x^3 + 2(c^6-2-17-2+61+28+3)x^2 + (-c^2+4+1)(c^3+1-5-1)^2(-c^4+2+20+34+9)x - P_2(c^3+1-5-1)^3,$$

$$(25)\quad F_3 = P_1x^3 + (-c^2+4+1)(c^7+1-23-67+35+219+83+7)x + 4(-c^2+4+1)(c^3+1-5-1)^2(-2c^5-1+22+44+16+1)x - Q_2(c^3+1-5-1)^3;$$

$$(26)\quad E_4 = P_1x^3 + 2(-c^2+4+1)(c^7+1-17-45+15+119+49+5)x^2 + (-c^2+4+1)(c^{13}-1-34-54+363+1037-588 - 4068-1881+4489+4238+1402+205+11)x + P_2(c^3+1-5-1)^4,$$

$$(27)\quad F_4 = x^4 + 2(c^6-2-21-6+73+32+3)x^3 + (c^{12}-4-46+68+743+264-3508-2952+5103 + 4988+1538+196+9)x^2 + (-c^2+4+1)(2c^{14}+5-71-234+300+1719 + 271-4732-3470+4227 + 5143+2054+384+33+1)x + Q^2(c^3+1-5-1)^4;$$

$$(28)\quad E_5 = x^4 + (3c^8 - 6 - 51 - 16 + 153 + 70 + 7)x^3 + (3c^{12} - 12 - 94 + 156 + 1241 + 408 - 4676 - 4152 + 5365 + 5876 + 1986 + 284 + 15)x^2 + (-c^2 + 4 + 1)(-c^{16} + 2 + 54 + 54 - 946 - 2590 + 2214 + 13798 + 4564 - 29290 - 26254 + 17842 + 28754 + 13190 + 2866 + 306 + 13)x - P_2[(c^3 + 1 - 5 - 1)^5 - 32c^3(c+1)^6(c-1)^4];$$

$$(29)\quad F_5 = P_1x^4 + 2(-c^2 + 4 + 1)(c^7 + 1 - 19 - 57 + 15 + 155 + 59 + 5)x^3 + (-c^2 + 4 + 1)(c^{13} - 1 - 42 - 86 + 483 + 1709 - 636 - 7364 - 3849 + 8937 + 7622 + 2170 + 261 + 11)x^2 + 4(-c^2 + 4 + 1)(-2c^{17} - 5 + 58 + 238 - 378 - 2554 + 602 + 13790 + 8130 - 30236 - 33282 + 15962 + 33682 + 17562 + 4462 + 602 + 40 + 1)x - Q_2[(c^3 + 1 - 5 - 1)^5 + 32c^3(c+1)^6(c-1)^4].$$

Arranged in powers of x,

$$(30)\quad R = R_0x^4 + R_1x^3 + R_2x^2 + R_3x + R_4,$$

$$(31)\quad S = S_0x^4 + S_1x^3 + S_2x^2 + S_3x + S_4,$$

$$(32)\quad R_0 = -1, \qquad S_0 = 5L - P_1,$$

. . . . . .

$$(33)\quad R_4 = P_1[(L + c^3 + c^2 - 5c - 1)^5 - 32c^3(c+1)^6(c-1)^4],$$
$$S_4 = Q_2[(L + c^3 + c^2 - 5c - 1)^5 + 32c^3(c+1)^6(c-1)^4].$$

In the upper rosette

$$(34)\quad L = \frac{Q}{x_2} = 4c, \quad M = (c-1)^3(c+1).$$

There are cusps on the circle $z = z_2$ when

$$(35)\quad (L - 4c)^3 = (c-1)^4(c+1)^2,$$

$$(36)\quad L = -c^3 + c^2 + 5c - 1 \quad \text{or} \quad c^3 - c^2 + 3c + 1;$$

and if the secular term pt is cancelled by putting

$$(37)\quad L = \tfrac{1}{5}(c+3)(-c^2 + 4c + 1),$$

$$(38)\quad (c-2)(c^2 + c - 1) = 0, \quad \text{or} \quad (3c-1)^3 + 10 = 0.$$

With

$$c = 2,\ L = 5,\ p = 0,\ M^2 = 24\sqrt{10}, \tag{39}$$

the motion of the top is given by

$$\begin{aligned}(\tfrac{1}{2}M^2\sin\theta\, e^{\phi i})^5 = (&-x^4 + 2^4\cdot 5\cdot 7x^3 - 2^4\cdot 3^4\cdot 5\cdot 19x^2 + 2^7\cdot 3^5\cdot 5^2 x \\ &- 2^7\cdot 3^8\cdot 5^2\cdot 23)\sqrt{(-x^2 + 80x + 8640)} \\ &+ i\cdot 2^6\cdot 5^2(x - 135)(x^2 - 2\cdot 3^3\cdot 5x + 2^4\cdot 3^5\cdot 5)\sqrt{(x - 135)},\end{aligned} \tag{40}$$

having 10 cusps at interval $\frac{1}{5}\pi$.

The other value $c = \frac{1}{3}(1 - \sqrt[3]{10})$ gives 10 cusps at interval $\frac{2}{5}\pi$; these curves are shown stereoscopically in L. M. S., vol. 27, p. 603.

In the lower rosette or intermediate curve

$$L = \frac{Q}{x_3 \text{ or } x_1} = -c^3 - c^2 + 3c + 1 \pm 2c\sqrt{(c^3 + c^2 - c)}, \tag{41}$$

and with cusps on $z = z_1$,

$$\left(L - \frac{Q}{x_1}\right)^2 = 4\sqrt{(c^3 + c^2 - c)}\,[c^2 + \sqrt{(c^3 + c^2 - c)}]^2. \tag{42}$$

25. The degree of the results for $I(v)$ can be halved by the substitution

$$y^2 = \frac{x}{x - x_2}, \tag{1}$$

and thereby the expression for the curve of H is simplified: but no reduction of degree is effected in the curve of P, because the functions of $\frac{1}{2}\chi$ in §11 are not simplified by this substitution.

Changing the limits in (2) §20,

$$\begin{aligned} I(v) &= \int_x^{x_3} \frac{-P(v)x + Q(v)}{x}\,\frac{dx}{\sqrt{X}} \\ &= \frac{1}{2n+1}\cos^{-1}\frac{x^n + Q_1x^{n-1} + \cdots + Q_n}{x^{n+\frac{1}{2}}}\sqrt{(x - x_2)} \\ &= \frac{1}{2n+1}\cos^{-1}\left(1 + \frac{Q_1}{x} + \cdots + \frac{Q_n}{x^n}\right)\sqrt{\left(1 - \frac{x_2}{x}\right)} \\ &= \frac{1}{2n+1}\cos^{-1}\left\{1 + \frac{Q_1}{x_2}\left(\frac{y^2-1}{y^2}\right) + \cdots + \frac{Q_n}{x_2^n}\left(\frac{y^2-1}{y^2}\right)^n\right\}\frac{1}{y} \\ &= \frac{1}{2n+1}\cos^{-1}\frac{R_0y^{2n} + R_1y^{2n-2} + \cdots}{y^{2n+1}} \end{aligned} \tag{2}$$

$$= \frac{2}{2n+1} \cos^{-1} \frac{y^{n-1} + A_1 y^{n-2} + \cdots + A_{n-1}}{y^{n+\frac{1}{2}}} \times \sqrt{\{\tfrac{1}{2}(y+1)(y^2 + C_1 y + C_2)\}}$$

$$= \frac{2}{2n+1} \sin^{-1} \frac{y^{n-1} - A_1 y^{n-2} + \cdots + (-1)^{n-1} A_{n-1}}{y^{n+\frac{1}{2}}} \times \sqrt{\{\tfrac{1}{2}(y-1)(y^2 - C_1 y + C_2)\}},$$

where the coefficients $A_1, \cdots, A_{n-1}$ are less than a quarter in number of the $P_1, \cdots, P_n$, $Q_1, \cdots, Q_n$, required previously.

Changing the variable from x to y in the integral $I(v)$, with the notation

$$y_1^2 = \frac{x_1}{x_1 - x_2} = \mathrm{sn}^2 fK', \tag{3}$$

$$y_3^2 = \frac{x_3}{x_3 - x_2} = \frac{1}{\mathrm{dn}^2(1-f)K'}, \tag{4}$$

$$Y = 4(y^2-1)(y^2-y_1^2)(y^2-y_3^2) = \frac{x_2^3}{(x_3-x_2)(x_2-x_1)} \cdot \frac{X}{(x-x_2)^4}, \tag{5}$$

$$\int_{y_3}^{y} \frac{dy^2}{\sqrt{Y}} = \sqrt{\frac{(x_3-x_2)(x_2-x_1)}{x_2}} \int_{x}^{x_3} \frac{dx}{\sqrt{X}} = \frac{\sqrt{\dfrac{-x_1x_3}{x_2}}}{y_1y_3} \int \frac{dx}{\sqrt{X}}, \tag{6}$$

$$I(v) = \int_{y_3}^{y} \frac{\left(\sqrt{\dfrac{-x_1x_3}{x_2}} - P\right)y^2 - \sqrt{\dfrac{-x_1x_3}{x_2}}}{y^2} \frac{y_1y_3}{\sqrt{\dfrac{-x_1x_3}{x_2}}} \frac{dy^2}{\sqrt{Y}} \tag{7}$$

$$= \int \frac{y_1y_3\sqrt{\dfrac{x_2}{-x_1x_3}}P(\omega_2 - v)y^2 - y_1y_3}{y^2} \frac{dy^2}{\sqrt{Y}}$$

$$= \int \frac{P'(v)y^2 - Q'(v)}{y^2} \frac{dy^2}{\sqrt{Y}}$$

$$P'(v) = y_1y_3\sqrt{\frac{x_2}{-x_1x_3}} \cdot P(\omega_2 - v), \quad Q'(v) = y_1y_3 = \frac{\mathrm{sn} fK'}{\mathrm{dn}(1-f)K'}. \tag{8}$$

Also we may take

$$y^2 \pm C_1 y + C_2 = (y \pm y_1)(y \mp y_3) \tag{9}$$

$$x_3 > x > x_2 > 0 > x_1, \tag{10}$$

$$x > y^2 > y_3^2 > 1 > y_1^2, \tag{11}$$

$$(12)\qquad \frac{P(v)}{\sqrt{(y_3^2-y_1)}} = \frac{P(\omega_2 - v)}{\sqrt{(x_3 - x_1)}} = \mathrm{zs}(1-f)K' = -\mathrm{zc}fK'.$$

26. Thus for $\mu = 6$, with this reduction,

$$(1)\qquad I(v) = \frac{2}{3}\cos^{-1}\sqrt{\}\frac{\frac{1}{2}(y+1)(y^2+ay-a)\{}{y^3}}$$

$$= \frac{2}{3}\sin^{-1}\sqrt{\}\frac{\frac{1}{2}(y-1)(y^2-ay-a)\{}{y^3}}$$

$$= \int \frac{\frac{1}{3}(1+a)y^2 - a}{y^2}\,\frac{dy^2}{\sqrt{Y}}.$$

Similarly, for $\mu = 10$,

$$(2)\quad I(v) = \frac{2}{5}\cos^{-1}\frac{y - \frac{2}{c+1}}{y^2}\sqrt{\left[\frac{1}{2}(y+1)\left\{y^2 + \frac{4c^2y + 4c}{(c+1)(c-1)^2}\right\}\right]}$$

$$= \frac{2}{5}\sin^{-1}\frac{y + \frac{2}{c+1}}{y^2}\sqrt{\left[\frac{1}{2}(y-1)\left\{y^2 + \frac{-4c^2y + 4c}{(c+1)(c-1)^2}\right\}\right]}$$

$$= \int \frac{\frac{c^3 - c^2 + 7c - 3}{5(c+1)(c-1)^2}y^2 - \frac{4c}{(c+1)(c-1)^2}}{y^2}\,\frac{dy^2}{\sqrt{Y}}.$$

The next case of $\mu = 14$ can now be attempted: utilising the analysis in L. M. S., vol. 25, p. 257, and vol. 27, p. 462, we can put

$$(3)\quad C_1 = \frac{(1-2m)(2-3m-\sqrt{M})}{4m(1-m)^3},\quad C_2 = \frac{(1-2m)^2(2-3m-\sqrt{M})}{4m(1-m)^3},$$

where

$$(4)\qquad M = 4m - 11m^2 + 8m^3,$$

$$(5)\qquad A_1 = -\frac{[\sqrt{(4-11m+8m^2)} - \sqrt{m}]^2}{4m(1-m)},$$

$$A_2 = \frac{(1-2m)[\sqrt{(4-11m+8m^2} - \sqrt{m}]^2}{4m(1-m)^2},$$

$$7P'(v) = \frac{-6 + 33m - 70m^2 + 64m^3 - 20m^4 + (3 - 6m + 4m^2)\sqrt{M}}{4m(1-m)^3},$$

where m is an arbitrary parameter.

The herpolhode curve of H can now be written down; but the associated motion of the top will be of very complicated character.

The result of $I(v)$ for $\mu = 18$ has been given in the *Archiv der Mathematik und Physik*, ser. 3, vol. 1, p. 73; and the preliminaries for $\mu = 22$ have been sketched out in the *Bulletin de la Société mathématique de France*, vol. 29, 1901, requisite for the calculation of C_1, C_2, $P'(v)$, A_1, A_2, A_3, A_4, in terms of a single parameter.

But the next cases of $\mu = 26$ and $\mu = 30$ still await solution.

27. With $\mu = 4n$, a further reduction is possible which makes the expression for the path of P or C' assume a symmetrical form, as was seen in the case discussed already, of

$$\mu = 4, \quad \text{and} \quad v = K + \tfrac{1}{2}K'. \tag{1}$$

Thus with $\mu = 8$, and

$$v = K + fK', \quad f = \tfrac{1}{4} \text{ or } \tfrac{3}{4}, \tag{2}$$

we shall find that, putting in the standard form of $I(v)$ in (1) §10,

$$s - s_3 = x^2, \quad \sigma - s_3 = \delta, \quad \sigma - s = \delta - x^2, \tag{3}$$

$$\begin{aligned} I(v) &= \int_x^k \frac{-P(v)(\delta - x^2) + Q(v)}{\delta - x^2} \frac{dx}{\sqrt{X_1 X_2}} \\ &= \frac{1}{2} \cos^{-1} \frac{\frac{1}{2}(1+a)^2 - x}{\delta - x^2} \sqrt{\tfrac{1}{2} X_1} \\ &= \frac{1}{2} \sin^{-1} \frac{\frac{1}{2}(1+a)^2 + x}{\delta - x^2} \sqrt{\tfrac{1}{2} X_2}, \end{aligned} \tag{4}$$

where

$$\begin{aligned} X_1 &= (1+x)\left\{\tfrac{1}{4}\left(\frac{1}{a} - a\right) + x\right\} = (1+x)(k+x), \\ X_2 &= (1-x)\left\{\tfrac{1}{4}\left(\frac{1}{a} - a\right)^2 - x\right\} = (1-x)(k-x), \end{aligned} \tag{5}$$

$$k = \tfrac{1}{4}\left(\frac{1}{a} - a\right)^2, \quad x = k \operatorname{sn}(K - mt); \tag{6}$$

as for $\mu = 4$; and the octahedron irrationality

$$o = \sqrt{k} = \tfrac{1}{2}\left(\frac{1}{a} - a\right), \tag{7}$$

unchanged by the substitution $\left(a, -\frac{1}{a}\right)$, which interchanges $f = \frac{1}{4}$ and $\frac{3}{4}$.

Also

$$\delta = \frac{(1+a)^3(1-a)}{4a}, \tag{8}$$

$$P(v) = \frac{(1+2a+3a^2)(-1+2a+a^2)}{16a^2}, \tag{9}$$

$$Q(v) = \frac{(1+a)^2(1-a^4)(-1+2a+a^2)}{16a^3}, \tag{10}$$

$$\mathrm{dn}^2 fK' = \frac{\sigma - s_3}{s_1 - s_3} = \delta, \tag{11}$$

$$\mathrm{sn}^2 fK' = \frac{4a^3}{(1+a^2)(1+2a-a^2)}, \quad \mathrm{cn}^2 fK' = \frac{(1+a)^3(1-a)}{(1+a^2)(1+2a-a^2)},$$

$$\mathrm{cn}\,2fK' = \frac{1-a^2}{1+a^2}, \quad \mathrm{dn}\,2fK' = o = \tfrac{1}{2}\left(\frac{1}{a} - a\right). \tag{12}$$

The vector of H is given by

$$M\frac{\rho}{c}\, e^{(pt-\omega)i} = [\tfrac{1}{2}(1+a)^2 - x\}\sqrt{\tfrac{1}{2}X_1} + i\}\tfrac{1}{2}(1+a)^2 + x\}\sqrt{\tfrac{1}{2}X_2}]^4, \tag{13}$$

$$\tfrac{1}{2}M^2 \sin\theta\, e^{\chi i} = \frac{L(\delta - x^2) - Q + ix\sqrt{X_1X_2}}{\sqrt{(\delta - x^2)}}, \tag{14}$$

$$(\tfrac{1}{2}M^2\sin\theta)^2\, e^{2(pt-\psi)i} \tag{15}$$

$$= \frac{[L(\delta - x^2) - Q + ix\sqrt{X_1X_2}]^2}{(\delta - x^2)^2}$$

$$\times [\tfrac{1}{2}(1+a)^2 - x\}\sqrt{\tfrac{1}{2}X_1} + i\}\tfrac{1}{2}(1+a)^2 + x\}\sqrt{\tfrac{1}{2}X_2}]$$

$$= (A_0 + A_1x + A_2x^2 + x^3)\sqrt{\tfrac{1}{2}X_1} + i(A_0 - A_1x + A_2x^2 - x^3)\sqrt{\tfrac{1}{2}X_2},$$

since the factor $(\delta - x^2)^2$ cancels; and we find

$$(16)\quad A_0 = \tfrac{1}{2}(1+a)^2\left\{L - \frac{(1+a^2)(-1+2a+a^2)}{4a^2}\right\}^2,$$

$$(17)\quad A_1 = -\left\{L - \frac{(1+a^2)(-1+2a+a^2)}{4a^2}\right\}\left\{L + \frac{1+2a+0-6a^3-a^4}{4a^2}\right\},$$

$$(18)\qquad A_2 = 2L - \tfrac{1}{2}(1+a)^2.$$

In a lower rosette

$$(19)\quad L = \frac{Q}{\delta} = \frac{(1+a^2)(-1+2a+a^2)}{4a^2},\ A_0 = 0,\ A_1 = 0,\ A_2 = -\tfrac{1}{2}\left(\frac{1-a}{a}\right)^2,$$

$$(20)\qquad M\cos\tfrac{1}{2}\theta = x, \qquad M^2 = \frac{(1+a)(1-a)^3}{4a^2},$$

$$(21)\quad \tan^2\tfrac{1}{2}\theta\, e^{2i(pt-\psi)i} = \frac{(A_2+x)\sqrt{\tfrac{1}{2}A_1} + i(A_2-x)\sqrt{\tfrac{1}{2}X_2}}{x^2}.$$

In an upper rosette

$$(22)\quad L = \frac{Q}{\delta - k^2} = a, \quad M^2 = \frac{(1-a)^2(1-a^4)(1+2a-a^2)}{16a^4},$$

$$M^2\sin^2\tfrac{1}{2}\theta = k^2 - x^2,$$

$$(23)\quad \cot^2\tfrac{1}{2}\theta\, e^{2i(pt-\psi)i} = \frac{\tfrac{1}{2}(1-a)^2 - x}{k-x}\sqrt{\left(\frac{1}{2}\,\frac{1+x}{k+x}\right)} + i\,\frac{\tfrac{1}{2}(1-a)^2 + x}{k+x}\sqrt{\left(\frac{1}{2}\,\frac{1-x}{k-x}\right)}.$$

In the intermediate case

$$(24)\quad L = \frac{Q}{\delta - 1} = -\frac{(1+a)^3(1-a)}{4a^2}, \quad M^2 = \frac{(1+a^2)(1+2a-a^2)}{4a^3},$$

$$M^2\sin^2\tfrac{1}{2}\theta = 1 - x^2,$$

$$(25)\quad \cot^2\tfrac{1}{2}\theta\, e^{2i(pt-\psi)i} = \frac{\frac{1}{2}\left(\frac{1+a}{a}\right)^2 - x}{1-x}\sqrt{\left(\frac{1}{2}\,\frac{k+x}{1+x}\right)} + i\,\frac{\frac{1}{2}\left(\frac{1+a}{a}\right)^2 + x}{1+x}\sqrt{\left(\frac{1}{2}\,\frac{k-x}{1-x}\right)}.$$

There are cusps on the circle $z = z_2$ when

$$(26) \qquad (L-a)^2 = \frac{(1-a)^2(1-a^4)(1+2a-a^2)}{16a^4};$$

and in the algebraical case when

$$(27) \qquad L = P, \quad L - a = -\frac{(1+2a-a^2)(1-2a+3a^2)}{16a^2},$$

$$(28) \qquad (1+2a-a^2)(1-2a+3a^2)^2 - 16(1-a)^2(1-a^4) = 0,$$

$$(29) \qquad 7a^6 - 2a^5 - 9a^4 + 12a^3 - 15a^2 + 30a - 15 = 0,$$

(L. M. S., vol. 27, p. 180), a sextic equation having two real roots

(30) $a = 0.642$, and -1.812, giving 8 cusps at interval $\frac{1}{4}\pi$, and $\frac{3}{4}\pi$,

as shown in the stereoscopic figure in L. M. S., vol. 27, p. 599.

28. We can now complete the case left unfinished in L. M. S., vol. 27, p. 605, of twelve branched gyroscope curves, given by

$$(1) \qquad \mu = 12, \quad v = K + fK'i, \quad f = \frac{1,5}{6};$$

$$(2) \qquad I = \int_x^{x_1} \frac{-P(\delta - x^2) + Q}{\delta - x^2} \frac{dx}{\sqrt{X_1 X_2}}$$

$$= \frac{1}{3}\cos^{-1}\frac{1+a+a^2+(1-a)x-x^2}{(a+a^2+a^3-x^2)^{\frac{3}{2}}}\sqrt{\tfrac{1}{2}X_1}$$

$$= \frac{1}{3}\sin^{-1}\frac{1+a+a^2-(1-a)x-x^2}{(a+a^2+a^3-x^2)^{\frac{3}{2}}}\sqrt{\tfrac{1}{2}X_2},$$

$$(3) \qquad X_1, X_2 = a^3(1+a+a^2) \pm \tfrac{1}{2}(1+a)^3(1-a)x - x^2,$$

$$(4) \qquad \delta = a + a^2 + a^3,$$

$$(5) \qquad P = \tfrac{1}{12}(1-a)(5+3a+3a^2+a^3),$$

$$(6) \qquad Q = \tfrac{1}{2}(1-a^4)(a+a^2+a^3);$$

$$(7)\qquad \left(M\frac{\rho}{c}\right)^3 e^{3(pt-\omega)i} = \{1 + a + a^2 + (1-a)\,x - x^2\}\sqrt{\tfrac{1}{2}X_1} + i\{1 + a + a^2 - (1-a)\,x - x^2\}\sqrt{\tfrac{1}{2}X_2},$$

$$(8)\qquad \frac{\rho^2}{c^2} = \frac{a + a^2 + a^3 - x^2}{M^2},$$

$$(9)\qquad x = x_2\,\mathrm{sn}(K - mt).$$

For the curve of P, working out the product of $(\frac{1}{2}M^2\sin\theta\, e^{\psi i})^3$ and $e^{3\theta i}$, the factor $(\delta - x^2)^3$ cancels, and

$$(10)\qquad (\tfrac{1}{2}M^2\sin\theta)^3 e^{3(pt-\psi)i} = (A_0 - A_1x + \cdots + A_4x^4 - x^5)\sqrt{\tfrac{1}{2}X_1} + i\,(A_0 + A_1x + \cdots + A_4x^4 + x^5)\sqrt{\tfrac{1}{2}X_2}.$$

I am indebted to Col. C. E. Basevi for the calculation of these A coefficients in this and the preceding case for $\mu = 8$: he finds

$$(11)\quad A_0 = (1 + a + a^2)L - \tfrac{1}{2}(1-a^4)\{^3,$$

$$A_1 = -\{L - \tfrac{1}{2}(1-a^4)\}\{^2\}(1-a)\,L - \tfrac{1}{2}(1 - a + 6a^2 + 6a^3 + 5a^4 + a^5)\{,$$

$$A_2 = -\{L - \tfrac{1}{2}(1-a^4)\}\{L^2 - (1-a)(4 + 4a + 4a^2 - a^3)L + \tfrac{1}{4}(7 + 0 + 0 - 6a^3 + 4a^4 + 12a^5 + 12a^6 + 6a^7 + a^8)\{,$$

$$A_3 = -3L^2 + 3(1-a)L - \tfrac{1}{4}(3 - 6a + 0 + 0 + 0 + 6a^5 + 4a^6 + 4a^7 + a^8),$$

$$A_4 = 3L - \tfrac{1}{2}(1-a)(5 + 3a + 3a^2 + a^3).$$

In the lower rosette

$$(12)\qquad L = \tfrac{1}{2}(1-a^4),\quad A_0 = A_1 = A_2 = 0,$$

$$A_3 = -a^6(1 + a + a^2),\quad A_4 = (1-a)(1+a^3),\quad M\cos\tfrac{1}{2}\theta = x,$$

$$(13)\qquad M^3\sin^3\tfrac{1}{2}\theta\, e^{3(pt-\psi)i} = x^3\tan^3\tfrac{1}{2}\theta\, e^{3(pt-\psi)i} = (-A_3 + A_4x - x^2)\sqrt{\tfrac{1}{2}X_1} + i(A_3 + A_4x + x^2)\sqrt{\tfrac{1}{2}X_2}.$$

For the upper rosettes and intermediate path the condition

$$(14)\qquad \left(L - \frac{Q}{\delta - x_1^2}\right)\left(L - \frac{Q}{\delta - x_2^2}\right) = 0$$

reduces to

$$(15)\quad L^2 - 1(1 - 4a - 4a^2 - 4a^3 - a^4)L - a(1 + a + a^2) = 0,$$
$$L = \tfrac{1}{4}\{1 - 4a - 4a^2 - 4a^3 - a^4 \pm (1 + a)^2\sqrt{(1 + a^2)(1 + 4a + a^2)}\}.$$

In fact

$$(16)\quad x_1,\ x_2 = \pm\tfrac{1}{4}(1 + a)^3(1 - a) + \tfrac{1}{4}(1 + a^2)\sqrt{(1 + a^2)(1 + 4a + a^2)},$$

$$(17)\quad o = \sqrt{k} = \sqrt{\frac{x_2}{x_1}} = \frac{-(1 + a)^3(1 - a) + (1 + a^2)\sqrt{A}}{4\sqrt{(a^3 + a^4 + a^5)}},$$
$$A = (1 + a^2)(1 + 4a + a^2).$$

$$(18)\quad \left(\frac{1}{o} - o\right)^2 = \frac{(1 + a)^6(1 - a)^2}{4a^3(1 + a + a^2)} = \frac{(b + 1)^3(b - 3)}{4b},$$

putting
$$a + 1 + \frac{1}{a} = b:$$

$$(19)\quad \left(\frac{1}{o} + o\right)^2 = \frac{(b - 1)^3(b + 3)}{4b},$$

$$(20)\quad \lambda^2 = \frac{4}{\left(\frac{1}{o} + o\right)^2} = \frac{16\,b}{(b - 1)^3(b + 3)},\quad \lambda'^2 = \left(\frac{b + 1}{b - 1}\right)^3\frac{b - 3}{b + 3}.$$

$$(21)\quad \mathrm{dn}^2 = fK'\ \frac{\delta}{x_1^2} = \frac{x_2^2}{a^4(1 + a + a^2)},\quad \frac{\mathrm{dn}\,fK'}{o} = \frac{1}{a}.$$

The cusp condition is

$$(22)\quad \left(L - \frac{Q}{\delta - x_2^2}\right)^2 = \frac{(x_1^2 - x_2^2)(x_2^2)}{\delta - x_2^2};$$

and becomes when rationalised in the algebraical case, when $L = P$, an equation of the 14th degree:

$$(23)\quad 875a^{14} + 2660 + 2765 + 280 - 4025 - 8900 - 10815 - 8930 - 4767 - 1124 + 295 + 280 + 173 + 68 + 11 = 0.$$

29. We are now prepared to generalise for μ a multiple of 4, and first we notice that we must distinguish the two cases of

$$(1)\quad \mu = 8n,\ \text{or}\ 8n + 4,$$

$$(2) \qquad v = K + \frac{rK'i}{4n}, \text{ or } K + \frac{rK'i}{4n+2};$$

$$(3) \qquad I = \int_x^{x_2} \frac{-P(\delta - x^2) + Q}{\delta - x^2} \frac{dx}{\sqrt{X_1 X_2}},$$

and according as $\frac{1}{4}\mu = 2n$, or $\frac{1}{4}\mu = 2n+1$,

$$(4) \qquad X_1 = P_0 + P_1 x + x^2, \quad \text{or} \quad P_0 + P_1 x - x^2,$$
$$= (x_1 + x)(x_2 + x) \text{ or } (x_1 - x)(x_2 + x),$$

$$(5) \qquad X_2 = P_0 - P_1 x + x^2, \quad \text{or} \quad P_0 - P_1 x - x^2,$$
$$= (x_1 - x)(x_2 - x), \text{ or } (x_1 + x)(x_2 - x),$$

$$(6) \qquad x_2 = kx_1, \quad \text{or} \quad x_1 = \frac{1}{o}, \; x_2 = o,$$

when normalised so as to make $P_0 = 1$;

$$(7) \qquad \delta = s(v) - s_3 = x_1^2 \operatorname{dn}^2 fK',$$

$$(8) \qquad P_0 = s_3 - s(2nv), \text{ or } \quad P_0 = s(2n+1)v - s_3,$$

$$(9) \qquad P_1 = 2P(2nv), \quad \text{or} \quad P_1 = 2P(2n+1)v,$$

$$(10) \qquad \left(\frac{1}{o} + o\right)^2 = \frac{P_1^2}{P_0}, \quad \text{or} \quad \left(\frac{1}{o} - o\right)^2 = \frac{P_1^2}{P_0},$$

$$(11) \qquad x = x_2 \operatorname{sn}(K - mt), \quad mt = \int_x^{x_2} \frac{x_1 dx}{\sqrt{X_1 X}}.$$

$$(12) \qquad I = \frac{1}{2n} \cos^{-1} \frac{R_0 + R_1 x + \cdots + R_{2n-1} x^{2n-1}}{(\delta - x^2)^n} \sqrt{\tfrac{1}{2} X_1},$$

$$\text{or } \frac{1}{2n+1} \cos^{-1} \frac{R_0 + R_1 x + \cdots + R_{2n} x^{2n}}{(\delta - x^2)^{n+\frac{1}{2}}} \sqrt{\tfrac{1}{2} X_1}$$

$$= \frac{1}{2n} \sin^{-1} \frac{R_0 - R_1 x + \cdots - R_{2n-1} x^{2n-1}}{(\delta - x^2)^n} \sqrt{\tfrac{1}{2} X_2},$$

$$\text{or } \frac{1}{2n+1} \sin^{-1} \frac{R_0 - R_1 x + \cdots + R_{2n} x^{2n}}{(\delta - x^2)^{n+\frac{1}{2}}} \sqrt{\tfrac{1}{2} X_2},$$

$$(13) \qquad R_{2n-1} = (-1)^n, \quad R_{2n} = (-1)^n.$$

The vector of H is then given by

$$(14)\qquad M\frac{\rho}{c}\,e^{(pt-\omega)i} = \sqrt{(\delta - x^2)}\,e^{Ji}$$

$$= \{(R_0 + R_1x + \cdots + R_{2n-1}x^{2n-1})\sqrt{\tfrac{1}{2}X_1}$$

$$+ i(R_0 - R_1x + \cdots - R_{2n-1}x^{2n-1})\sqrt{\tfrac{1}{2}X_2}\}^{\frac{1}{2n}},$$

$$\text{or}\quad \{(R_0 + R_1x + \cdots + R_{2n}x^{2n})\sqrt{\tfrac{1}{2}X_1}$$

$$+ i(R_0 - R_1x + \cdots + R_{2n}x^{2n})\sqrt{\tfrac{1}{2}X_2}\}^{\frac{1}{2n+1}}$$

and multiplying $\frac{1}{2}M^2$ in θe^{xi} by e^{Ji}, rationalising, and cancelling the factor

$$(\delta - x^2)^{2n,\text{ or }2n+1},$$

$$(15)\qquad \{\tfrac{1}{2}M^2\sin\theta\, e^{(pt-\phi)i}\}^{2n,\text{ or }2n+1}$$

$$= (A_0 + A_1x + \cdots + A_{4n-1}x^{4n-1})\sqrt{\tfrac{1}{2}X_1}$$

$$+ i(A_0 - A_1x + \cdots - A_{4n-1}x^{4n-1})\sqrt{\tfrac{1}{2}X_2},$$

$$\text{or } (A_0 + A_1x + \cdots + A_{4n+1}x^{4n+1})\sqrt{\tfrac{1}{2}X_1}$$

$$+ i(A_0 - A_1x + \cdots - A_{4n+1}x^{4n+1})\sqrt{\tfrac{1}{2}X_2};$$

where the determination of the coefficients R and A is a piece of algebraical labour of increasing scale as μ increases.

30. Putting in the standard form of (1) §10

(1) $\sigma - s_1,\ \sigma - s_2,\ \sigma - s_3 = \sigma_1, \sigma_2, \sigma_3$ (denoted previously by x_1, x_2, x_3),

$$(2)\qquad \sigma_1 = \delta - x_1^2,\ \sigma_2 = \delta - x_2^2,\ \sigma_3 = \delta,$$

$$(3)\qquad 3\varphi v = \sigma_1 + \sigma_2 + \sigma_3 = 3\delta - x_1^2 - x_2^2 = x_1^2(1 + k'^2 - 3k'^2\,\mathrm{sn}^2 fK'),$$

$$(4)\qquad -\tfrac{1}{2}i\varphi' v = Q = \sqrt{(-\sigma_1, \sigma_2, \sigma_3)} = \sqrt{(x_1^2 - \delta)(\delta - x_2^2)(\delta)}$$

$$= x_1^3\,k'^2\,\mathrm{sn}\,fK'\,\mathrm{cn}\,fK'\,\mathrm{dn}fK',$$

$$(5)\qquad \tfrac{1}{2}\varphi'' v = \sigma_2\sigma_3 + \sigma_1\sigma_1 + \sigma_1\sigma_2 = 3\delta^2 - 2(x_1^2 + x_2^2)\,\delta + x_1^2 x_2^2,$$

$$(6)\qquad M^2 z = L^2 + \delta - x_1^2 - x_2^2 + 2x^2,$$

$$(7)\qquad \tfrac{1}{2}M^2(E - z) = \sigma - s = \delta - x^2,$$

$$(8)\qquad \tfrac{1}{2}M^2(z - z_3) = s - s_3 = x^2,$$

$$(9)\qquad \tfrac{1}{2}M^2(z_2 - z) = s_2 - s = x_2^2 - x^2,$$

$$(10)\qquad \tfrac{1}{2}M^2(z_1 - z) = s_1 - s = x_1^2 - x^2.$$

From (7) §11,

$$(11)\quad \tfrac{1}{4}M^4\sin^2\theta = -2LQ - \tfrac{1}{2}\wp''(v) + (L^2+3\wp v)(\sigma-s) - (\sigma-s)^2$$
$$= [L\sqrt{\delta}\sqrt{(x_1^2-\delta)(\delta-x_2^2)}]^2 - (L^2+\delta-x_1^2-x_2^2)x^2 - x^4,$$

$$(12)\quad M^4 = (L^2+\delta-x_1^2-x_2^2)^2 + 4[L\sqrt{\delta} - \sqrt{(x_1^2-\delta)(\delta-x_2^2)}]^2,$$

$$(13)\quad \tfrac{1}{2}M^2\sin\theta_3 = L\sqrt{\delta} - \sqrt{(x_1^2-\delta)(\delta-x_2^2)}.$$

In lower rosettes, writing f for fK',

$$(14)\quad L\sqrt{\delta} = \sqrt{(x_1^2-\delta)(\delta-x_2^2)},\ L = x_1k'^2\frac{\operatorname{sn}f\operatorname{cn}f}{\operatorname{dn}f},\ M\cos\tfrac{1}{2}\theta = x,$$

$$(15)\quad M^2\sin^2\tfrac{1}{2}\theta = -L^2-\delta+x_1^2+x_2^2-x^2,$$

$$(16)\quad M^2 = -L^2-\delta+x_1^2+x_2^2 = \frac{x_1^2x_2^2}{\delta},\ M = x_1\operatorname{dn}(1-f)K'.$$

In upper rosettes (or intermediate path),

$$(17)\quad L\sqrt{(\delta-x_2^2)} = \sqrt{(x_1^2-\delta)\delta},\quad L = x_1\frac{\operatorname{sn}f\operatorname{dn}f}{\operatorname{cn}f}\left(\text{or } x_1\frac{\operatorname{cn}f\operatorname{dn}f}{\operatorname{sn}f}\right),$$

$$(18)\quad \tfrac{1}{2}M^2(1-z) = x_2^2 - x^2 (\text{or } x_1^2-x^2),$$

$$(19)\quad \tfrac{1}{2}M^2(1+z) = L^2+\delta-x_1^2+x^2 = \frac{x_1^2-\delta}{\delta-x_2^2}x_2^2+x^2,$$

$$(20)\quad M^2 = x_2^2\frac{x_1^2-x_2^2}{\delta-x_2^2},\ M = \frac{kx_1}{\operatorname{cn}f}\text{ or }\frac{x_1}{\operatorname{sn}f}.$$

For cusps on the circle $z = z_2$ (or z_1)

$$(21)\quad \left[L - \sqrt{\left(\frac{x_1^2-\delta}{\delta-x_2^2}\delta\right)}\right]^2 = \frac{x_1^2-x_2^2}{\delta-x_2^2}x_2^2,$$

$$(22)\quad L = \frac{Q + x_2\sqrt{(x_1^2-x_2^2)(\delta-x_2^2)}}{\delta-x_2^2}$$
$$= x_1\frac{\operatorname{sn}f\operatorname{dn}f+k}{\operatorname{cn}f}.$$

The calculations can now be carried out for $\mu = 16, 24, 32, 40$, and $\mu = 20, 28, 36, 44$; but $\mu = 52$ and 60 must await the reduction of $\mu = 26$ and 30, not yet accomplished.

31. The chief practical interest is satisfied by the determination of the motion of the axis of the top, by Euler's angles θ and ψ; but to complete the solution theoretically we must determine Euler's third angle ϕ, and thence Klein's $\alpha, \beta, \gamma, \delta$.

If v_1, v_2, denote the values of u corresponding to $z = -1, 1$, then since

$$z_1 > 1 > z_2 > z > z_3 > -1, \tag{1}$$

we must take

$$v_1 = f_1\,\omega_3, \quad v_2 = \omega_1 + f_2\,\omega_3, \tag{2}$$

where f_1, f_2 are real fractions; and v, w corresponding to $z = E$, D.

$$v = v_1 - v_2, \quad w = v_1 + v_2. \tag{3}$$

In the spherical top in which $A_1 = C_1$, which we may take without essential loss of generality,

$$\frac{d\phi}{dt} = R - \cos\theta\frac{d\psi}{dt} = \frac{G'}{A_1} - \cos\theta\,\frac{d\psi}{dt}, \tag{4}$$

$$\frac{d\psi}{dt} = \frac{G - G'\cos\theta}{A_1\sin^2\theta}, \tag{5}$$

$$\frac{d\phi}{dt} = \frac{G' - G\cos\theta}{A_1\sin^2\theta}, \tag{6}$$

$$\frac{d(\phi+\psi)}{dt} = \frac{G' + G}{A_1(1+\cos\theta)}, \quad \phi + \psi = 2\,\frac{L' + L}{M}\int_z^{z_1}\frac{dz}{(1+z)\sqrt{2Z}}, \tag{7}$$

$$\frac{d(\phi-\psi)}{dt} = \frac{G' - G}{A_1(1-\cos\theta)}, \quad \phi - \psi = 2\,\frac{L' - L}{M}\int\frac{dz}{(1-z)\sqrt{2Z}}. \tag{8}$$

Now with

$$\tfrac{1}{2}M^2(1+z) = M^2\cos^2\tfrac{1}{2}\theta = s - s(v_1), \tag{9}$$

$$\tfrac{1}{2}M^2(1-z) = M^2\sin^2\tfrac{1}{2}\theta = s(v_2) - s, \tag{10}$$

$$M^2 = s(v_2) - s(v_1), \tag{11}$$

$$\wp' u = S = \tfrac{1}{2}M^6 Z, \tag{12}$$

$$-S(v_1) = M^4(L' + L)^2, \quad -S(v_2) = M^4(L' - L)^2. \tag{13}$$

Then

$$\tfrac{1}{2}(\phi + \psi) = \int_s^{s_2} \frac{\tfrac{1}{2}\sqrt{-S(v_1)}}{s - s(v_1)} \frac{ds}{\sqrt{S}} \tag{14}$$

$$\tfrac{1}{2}(\phi - \psi) = \int \frac{\tfrac{1}{2}\sqrt{-S(v_2)}}{s(v_2) - s} \frac{ds}{\sqrt{S}} \tag{15}$$

or

$$\tfrac{1}{2}(\phi + \psi) = \frac{P(v_1)}{M} nt - I(v_1) = p_1 t - I(v_1), \tag{16}$$

$$\tfrac{1}{2}(\phi - \psi) = \frac{P(v_2)}{M} nt + I(v_2) = p_2 t + I(v_2), \tag{17}$$

where

$$I(v_1) = \int_s^{s_2} \frac{+ P(v_1)[s - s(v_1)] - \tfrac{1}{2}\sqrt{-S(v_1)}}{s - s(v_1)} \frac{ds}{\sqrt{S}}, \tag{18}$$

$$I(v_2) = \int \frac{-P(v_2)[s(v_2) - s] + \tfrac{1}{2}\sqrt{-S(v_2)}}{s(v_2) - s} \frac{ds}{\sqrt{S}}, \tag{19}$$

$$I(v) = \tfrac{1}{2}\, i \log \frac{\theta(u + v)}{\theta(u - v)}. \tag{20}$$

Now, introducing Klein's functions, α, β, γ, δ, cited at the outset,

$$M\alpha = \sqrt{s - s(v_1)}\, e^{[p_1 t - I(v_1)]\, i}$$

$$= e^{p_1 t i}\, \frac{\theta(0)\theta(u + v_1)}{\lambda\, \theta u\, \theta v_1} = e^{p_1 t i}\, (A_1 + B_1 i), \tag{21}$$

suppose; and

$$M\delta = e^{-p_1 t i}\, \frac{\theta(0)\theta(u + v_1)}{\lambda\, \theta u\, \theta v_1} = e^{-p_1 t i}\, (A_1 - B_1 i), \tag{22}$$

with

$$M^2 \alpha\delta = s - s(v_1) = A_1^2 + B_1^2. \tag{23}$$

Also

$$M\gamma = i\sqrt{s(v_2) - s}\, e^{[p_2 t + I(v_2)]\, i}$$

$$= i e^{p_2 t i}\, \frac{\theta(0)\theta(u - v_2)}{\lambda\, \theta u\, \theta v_2} = e^{p_2 t i}\, (i\, A_2 - B_2\, i), \tag{24}$$

$$M\beta = i\, e^{-p_2 t i}\, \frac{\theta(0)\theta(u + v_2)}{\lambda\, \theta u\, \theta v_2} = e^{-p_2 t i}\, (i\, A_2 + B_2\, i), \tag{25}$$

$$M^2\beta\gamma = s - s(v_2) = -A_2^2 - B_2^2, \tag{26}$$

$$M^2(\alpha\delta - \beta\gamma) = s(v_2) - s(v_1) = M^2 = A_1^2 + B_1^2 + A_2^2 + B_2^2. \tag{27}$$

The multiplicative elliptic functions α, β, λ, δ are Lamé functions of the first order, as they are found to satisfy the differential equations

$$\frac{1}{\alpha}\frac{d^2\alpha}{du^2}, \quad \text{or} \quad \frac{1}{\delta}\frac{d^2\delta}{du^2} = 2\wp u + \wp v_1, \tag{28}$$

$$\frac{1}{\beta}\frac{d^2\beta}{du^2}, \quad \text{or} \quad \frac{1}{\gamma}\frac{d^2\gamma}{du^2} = 2\wp u + \wp v_2; \tag{29}$$

this follows from their expression by means of (9), (10), (14), (15), in the Weierstrass notation

$$M\alpha = e^{-u\zeta v_1}\frac{\sigma(u + v_1)}{\sigma u \sigma v_1}, \text{ etc.} \tag{30}$$

Also (2) §3 may be written

$$2A_1 n^2 \alpha\gamma = \frac{d}{dt}(\rho e^{\omega i}). \tag{31}$$

32. With lower rosettes

$$L' + L = 0, \quad G' + G = 0, \quad \phi + \psi = 0, \quad f_1 = 1; \tag{1}$$

$$\alpha = \delta = \cos\tfrac{1}{2}\theta, \tag{2}$$

$$\beta = i\sin\tfrac{1}{2}\theta\, e^{\psi i}, \quad \gamma = i\sin\tfrac{1}{2}\theta\, e^{-\psi i}. \tag{3}$$

With upper rosettes or intermediate paths,

$$L' - L = 0, \quad G' - G = 0, \quad \phi - \psi = 0, \quad f_2 = 1, \text{ or } 0, \tag{4}$$

$$\beta = \gamma = i\sin\tfrac{1}{2}\theta, \tag{5}$$

$$\alpha = \cos\tfrac{1}{2}\theta\, e^{\psi i}, \quad \delta = \cos\tfrac{1}{2}\theta\, e^{-\psi i}. \tag{6}$$

With cusps on the circle $z = z_2$ or z_1, writing

$$Z = (1 - z^2)(D - z) - \frac{1}{2}\left(\frac{G - G'z}{An}\right)^2 = (1 - z^2)(D - z) - 2\left(\frac{L - L'z}{M}\right)^2, \tag{7}$$

then $G - G'z$ and $D - z$ are factors of Z,

$$\frac{G}{G'} = \frac{L}{L'} = D = z_2 \text{ or } z_1, \tag{8}$$

$$w = \omega_2 \text{ or } \omega_1, \quad f_1 + f_2 = 1 \text{ or } 0. \tag{9}$$

Now in the associated curve (ρ', ω') of H in the plane HKC'

$$\omega' = p't, \quad \phi = pt - \chi'; \tag{10}$$

$$\tfrac{1}{2}M^2 \sin\theta\, e^{\chi i} = L' \sqrt{x'} + i\sqrt{(x'_3 - x')(x' - x'_1)}, \quad x' = \tfrac{1}{2}M^2(D - z), \tag{11}$$

and the spherical curve of C' is rectifiable, in the form

$$\frac{s}{c} = \int_{z_3}^{z} \frac{dz}{\sqrt{(z_1 - z)(z - z_3)}} = 2\tan^{-1}\sqrt{\frac{z - z_3}{z_1 - z}} \tag{12}$$

$$= 2\tan^{-1}(\operatorname{cn} mt) \text{ or } 2\operatorname{am}(K - mt),$$

when z_1 and z_2 change places.

33. When f_2 is a rational fraction, so that μv_2 is congruent to a period, we can utilize the preceding results, and express $(A_2 + B_2 i)$ by algebraical functions of z; and now the μth powers of β and γ are algebraical functions qualified by exponential functions of the time.

If we wish to construct a case of motion in which α and δ are quasi-algebraical at the same time it will be sufficient for illustration to take $f_1 = 2f_2$, so that $v_1 = 2v_2$, and now $I(2v)$ is deduced very easily from $I(v)$.

As an illustration, take

$$v_1 = \tfrac{1}{2}\omega_3, \qquad v_2 = \omega_1 + \tfrac{1}{4}\omega_3, \tag{1}$$

so that from (2) §18, (4) §27,

$$M\gamma = i\, e^{p_2 ti}\left[\tfrac{1}{2}(1 + a)^2 - x,\ \tfrac{1}{2}\sqrt{X_1} + i\ \tfrac{1}{2}(1 + a)^2 + x,\ \sqrt{\tfrac{1}{2}X_2}\right], \tag{2}$$

$$M^2 \sin^2\tfrac{1}{2}\theta = \frac{(1 + a)^3(1 - a)}{4a} - x^2, \tag{3}$$

$$M\alpha = p^{p_1 ti}\left[\sqrt{\tfrac{1}{2}X_1} + i\sqrt{\tfrac{1}{2}X_2}\right], \tag{4}$$

$$M^2 \cos^2\tfrac{1}{2}\theta = k + x^2, \tag{5}$$

$$X_1,\ X_2 = (1 \pm x)(k \pm x), \quad k = \tfrac{1}{4}\left(\frac{1}{a} - a\right)^2, \tag{6}$$

$$M^2 = \frac{(1 + a)^3(1 - a)}{4a} + k = \frac{(1 + a)(1 - a^4)}{4a^2}. \tag{7}$$

With

$$v_1 = \tfrac{2}{3}\omega_3, \quad v_2 = \omega_1 + \tfrac{1}{3}\omega_3, \quad w = \omega_1 + \omega_3, \tag{8}$$

there are cusps on the circle $z = z_2$; and from (4) §19,

$$(9)\quad M\gamma = i\,e^{p_1 d i}\left[\tfrac{1}{2}(9-b^2)\sqrt{(x_3-x)(x-x_1)} + i(x+9-b^2)\sqrt{(x-x_2)}\right]^{\frac{1}{3}},$$

$$(10)\quad P(v_2) = \tfrac{1}{6}(9-b^2), \quad M^2\beta\gamma = s - s(v_2) = -x;$$

$$(11)\quad Ma = e^{p_1 d i}\left[\tfrac{1}{2}(b^2+3)x - \tfrac{1}{2}(b^2-1)(b^2+7) + i\sqrt{\tfrac{1}{4}X}\right]^{\frac{1}{3}},$$

$$(12)\quad P(v_2) = \tfrac{1}{6}(b^2+3), \quad M^2\alpha\delta = 2(b^2-1) - x, \quad M^2 = 2(b^2-1).$$

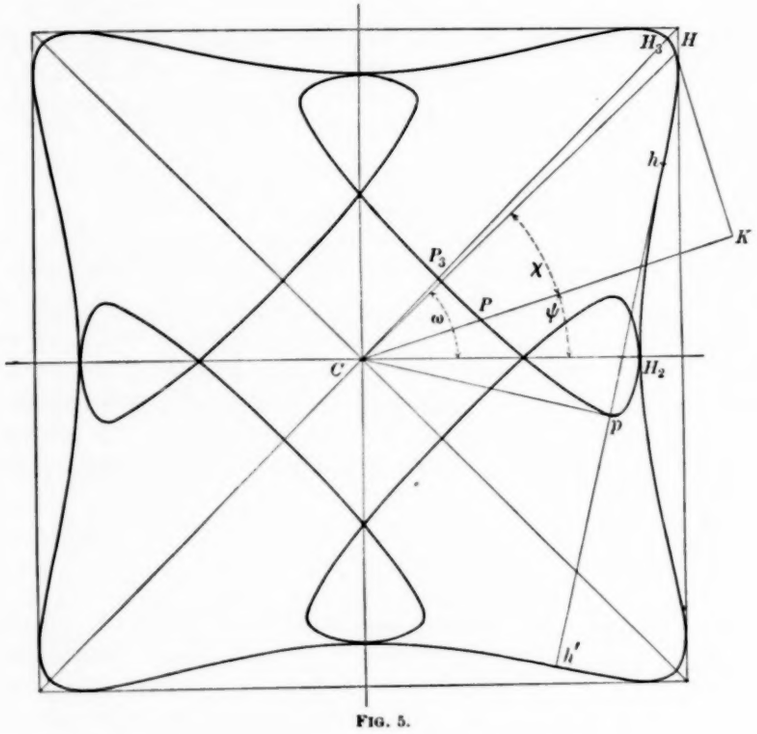

FIG. 5.

With (L.M.S., vol. 25, p. 239 and the results of §24),

$$(13)\quad v_1 = \tfrac{2}{5}\omega_3, \quad v_2 = \omega_1 + \tfrac{1}{5}\omega_3,$$

$$(14)\quad M\gamma = ie^{p_1 d i}\left[(P_1 x + P_2)\sqrt{(x_3-x)(x-x_1)} + i(x^2 + Q_1 x + Q_2)\sqrt{(x-x_2)}\right]^{\frac{1}{3}},$$

$$(15)\quad P(v_2) = \tfrac{1}{5}(c+3)(-c^2+4c+1), \quad M^2\beta\gamma = -x;$$

$$(16)\quad Ma = e^{p_1 ti}\left[R_0 x^2 + R_1 x + R_2 + i\right\}x - 4(c-1)(c^3+7c^2-c+1)\left\{\sqrt{\tfrac{1}{4}X}\right]^{\frac{1}{2}},$$

$$(17)\quad P(v_1) = \tfrac{1}{5}(c^3 - c^2 + 7c - 3), \quad M^2 a\delta = 8c(c+1)^2(c-1) - x_1,$$
$$M^2 = 8c(c+1)^2(c-1);$$

$$(18)\quad R_0 = c^3 - c^2 + 7c - 3,$$
$$R_1 = 4(c-1)(-2c^6 - 7 - 23 - 46 + 8 + 5 + 1),$$
$$R_2 = 16c(c+1)^3(c-1)^2(c^5 + 3c^4 + 34c^3 - 2c^2 - 3c - 1).$$

With the original s of the standard form of $I(v)$ in (1) §10,

$$(19)\quad s = 8c(c+1)^2(c-1) - x,$$

$$(20)\quad Ma = e^{p_1 ti}\ \left[(c^3 - c^2 + 7c - 3)\,s^2 - 4(c-1)^4(2c^3 + 3c^2 + 4c - 1)s\right.$$
$$\left. + 16c(c-1)^8(c+1)^2 + i\right\}s - 4(c-1)^4\left\{\sqrt{\tfrac{1}{4}S}\right]^{\frac{1}{2}}$$

$$(21)\quad M^2 a\delta = \tfrac{1}{2}M^2(1+\cos\theta) = s;$$

and putting $s = x^2$, we can halve the degree of Ma by writing it

$$(22)\quad Ma = e^{p_1 ti}\ \left[\right\}x + 2(c-1)^2\left\{\sqrt{\tfrac{1}{2}S_1} + i\right\}x - 2(c-1)^2\left\{\sqrt{\tfrac{1}{2}S_2}\right]^{\frac{1}{2}}.$$

34. The geometrical interpretation of the herpolhode and associated motion of the top, by means of Poinsot's rolling quadric and Darboux's deformable hyperboloid is required for the complete treatment of the problem, but this would lead too far at present. *Nature*, Aug., 1899, may be consulted for a slight sketch.

LONDON, MARCH, 1903.

ADDENDA AND ERRATA.

Page 7. Equation (12) § 7 is equivalent to

$$\frac{OK^2 = \delta^2 - 2\delta\delta'\cos\theta + \delta'^2}{\sin^2\theta} = \frac{CC'^2}{\sin^2\theta},$$

as is evident geometrically, so that the characteristic property of the polhode line of curvature depends otherwise, on equation (2), and the constancy of the lengths δ, δ' cut off from the generating lines of the confocal hyperboloid of one sheet through a line of curvature on an ellipsoid, by perpendicular planes through the centre.

Page 18. Line 2, add — "with angular velocity $2n\mathrm{sh}\,\frac{1}{2}\theta_1 \sin\frac{1}{2}\theta_3$.

Page 18. Line 12, add — "with angular velocity $2n\mathrm{ch}\,\frac{1}{2}\theta_1 \cos\frac{1}{2}\theta_3$."

Page 20. Line 2, read "$= L(L^2 + A)^2 - 8L^2Q - 4LB$."

A RELATION BETWEEN THE CIRCULAR AND THE PROJECTIVE TRANSFORMATIONS OF THE PLANE*

By Edward Kasner

1. Introduction. Among the transformations of the plane, those which play the most important rôle in modern geometry are the two mentioned in the title.† The projective transformations (otherwise homographies or collineations) are characterized among point transformations by the property that straight lines are converted into straight lines; and are represented, in Cartesian coordinates, as follows:

$$(H) \qquad x' = \frac{a_1 x + b_1 y + c_1}{a_3 x + b_3 y + c_3}, \qquad y' = \frac{a_2 x + b_2 y + c_2}{a_3 x + b_3 y + c_3};$$

thus constituting a continuous group of eight parameters. The circular transformations, on the other hand, convert circles into circles, and are represented most conveniently in terms of the complex variable. They divide into two species, those of the first taking the form

$$(\Gamma_1) \qquad z' = \frac{\alpha z + \beta}{\gamma z + \delta} \qquad (z = x + iy),$$

while those of the second (distinguished by the reversal of angles) take the form

$$(\Gamma_2) \qquad z' = \frac{\alpha \bar{z} + \beta}{\gamma \bar{z} + \delta} \qquad (\bar{z} = x - iy).$$

The totality of circular transformations constitutes then a mixed group of six real parameters, composed of two continuous systems.

The simple relation established in the present paper shows how either type of transformation may be obtained from the other type. It is of course useless to seek for a (continuous) one to one correspondence between the two classes, since the number of parameters involved is not the same. But the effect of the relation in question may be roughly described as the establishment of

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* Read before the American Mathematical Society, February 28, 1903.

† See, for example, Klein's *Erlanger Programm* and *Höhere Geometrie*.

a correspondence between the homographies (H) on the one hand, and, on the other, the combinations (Γ, O), each composed of a circular transformation and a point.

The correspondence, however, breaks down for certain special transformations — a circumstance which is to be expected in view of the fact that in projective geometry the infinite region of the plane is regarded as a straight line, while in inversion geometry it is equivalent to a point. The exceptional transformations are indeed those which convert the infinite region into itself. We shall assume then in the following, unless stated otherwise, that H is not an affinity, and Γ is not a similarity. Consequently H has in each plane two finite points, the first and the second focus, the centres respectively of the corresponding congruent and symmetric pencils of rays;* and Γ has, in each plane, a unique finite point, the pole, which corresponds to the infinite point in the other plane.†

2. From Circular to Projective Transformations. We proceed now to the relation in question. Consider a circular transformation Γ and a finite point O which does not coincide with the pole of Γ. Denote the corresponding point, necessarily finite, in the second plane, by O'. Any circle through O is converted by Γ into a circle through O'. But to each of these circles there corresponds a unique point, its center. In this way, we obtain a certain transformation, H, not coinciding with Γ, whose effect upon any point P is described as follows: Construct the circle γ passing through O and having P for center; this is converted by Γ into a circle γ' passing through O'; the center of γ' is the point P' required.

We now prove that the transformation H, so obtained, is a homography, by showing that any straight line is converted into a straight line. Consider any line l. The circles passing through O and having their centers on l, form a pencil. These circles are converted by Γ into circles through O' which also form a pencil. Therefore the centers lie on a line l'. This proves the result stated.

The construction for l' may be described more simply as follows: Construct the image of O with respect to l, say P; next the point P' into which P

* H. F. S. Smith, The focal properties of homographic figures, § 5, *Proc. Lond. Math. Soc.* (1869) or *Collected Mathematical Papers*, vol. 1, p. 549. According to the historical note at the end of this memoir, Magnus and Chasles in their investigations considered only the first focus.

† Möbius, Theorie der Kreisverwandtschaften in rein geometrischer Darstellung, §6, *Gesammelte Werke*, vol. 2.

is converted by Γ; finally the mid-line of the points O' and P'; this is the required line l'.

This construction fails, however, if the line l passes through O; but in this case the line l' passes through O', and is easily obtained by taking any circle γ tangent to l at O; the line l' is then tangent to the circle γ' into which Γ converts γ. It follows that the angle of any two lines passing through O, is equal to the angle of the corresponding lines through O', being reversed in direction however if Γ is of the second species. Hence the point O is a focus of H. The result obtained may be stated:

THEOREM I. *To a circular transformation Γ, and a point O which is not its pole, corresponds a unique collineation H. The point O is the first or second focus of H according as Γ is of the first or second species.**

3. From Projective to Circular Transformations. The H obtained of course depends, in general, upon both Γ and O; so that a single circular transformation gives rise to ∞^2 collineations,† corresponding to the ∞^2 positions of O. It seems plausible that we obtain, by the above construction, the totality of collineations, or at least all those possessing foci.‡ The exact theorem follows:

THEOREM II. *To any homography H which is not an affinity, there correspond two combinations (Γ, O), composed of a circular transformation and a point, each of which gives rise to H. In one of these, (Γ_1, F_1), the transformation is of the first species, and the point is the first focus of H; while in the other, (Γ_2, F_2), the transformation is of the second species and the point is the second focus of H.*

For the proof, we note first that the point O must be a focus of H. Consider then either focus F, and the corresponding point F' in the other plane. Take any three points A, B, C, such that no three of the four points F, A, B, C are collinear; and let the corresponding points be A', B', C'. Consider the circles α, β, γ passing through F and having A, B, C for centers; and similarly the circles α', β', γ' passing through F' and with centers at

* For the exceptional case noted in the introduction, when Γ is a similarity S, the first part of the result still holds. The homography obtained is then simply S. Even the second part may be said to hold, since in the case of a similarity any point has the property of a focus.

† It may be remarked that these do *not* form a group.

‡ In the case of any affinity no focus exists, except when it is merely a similarity S; in the latter case, any point may be considered as a focus, so that the corresponding combinations are (S, O), where O is any point.

A', B', C'. We have thus in each plane a set of three circles of a bundle. Since the angle of any two circles of either bundle is equal to the angle between the lines drawn from the focus to the centers, it follows that corresponding angles in the two bundles are equal. Hence there exists a unique circular transformation Γ which converts α, β, γ into α', β', γ' respectively. This transformation Γ together with the point F' gives rise, by Theorem I, to a unique homography. It is easily seen that the homography so obtained converts F, A, B, C into F', A', B', C', and therefore coincides with the given collineation H. This completes the proof of (II).

The effect of Γ may be described as follows: Consider any point P; construct the mid-line l of F and P; next the line l' into which l is converted by H; finally the image of O' with respect to l': this is the point P' which corresponds, by Γ, to the original point P.

4. Analytical Representation of the Relation. By a proper choice of the system of axes in each plane, it is possible to reduce the general collineation H to the following metrically normal form:*

$$x' = \frac{cc'}{x}, \quad y' = \frac{c'y}{x}. \tag{1}$$

The origin in each plane is the point midway between the two foci, and the axis of abscissas is the line joining the foci; the axis of ordinates is then the vanishing line. In the first plane the foci are

$$F_1: \quad (-c, 0); \qquad F_2: \quad (c, 0). \tag{2}$$

The corresponding circular transformations are obtained by following out, step by step, the construction given at the end of the preceding section; the results are:

$$\Gamma_1: \quad z' = c' \frac{z + 3c}{z - c}, \tag{3}$$

$$\Gamma_2: \quad z' = -c' \frac{\bar{z} - 3c}{\bar{z} + c}. \tag{4}$$

We note from these normal forms, that *the pole of* Γ_1 *is* F_2 *the second focus of* H, *and the pole of* Γ_2 *is* F_1 *the first focus of* H. The coordinate

* *Cf.* §15 of Smith's memoir, cited above.

system may therefore be defined anew, with reference to the combination (Γ, F), as follows: the origin is the point midway between F and the pole of Γ, and the axis of abscissas the line joining the same points.

We now inquire concerning those points upon which H and Γ have the same effect. Considering in the first place Γ_1, we find from (1) and (3) that the points in question satisfy the condition

$$\frac{c'(c+iy)}{x} = \frac{c'(x+iy+3c)}{x+iy-c},$$

which reduces to

$$(x+c)^2 + y^2 = 0.$$

This represents the null circle whose center is F. Similarly, comparing (4) and (1), we obtain

$$\frac{c'(c+iy)}{x} = \frac{-c'(x-iy-3c)}{x-iy+c},$$

which reduces to

$$(x-c)^2 + y^2 = 0,$$

and represents the null circle with center at F_2.

If the homography H corresponds to the combination (Γ, O), then the only real point upon which H and Γ have the same effect is the point O. The imaginary points having the property in question lie on the minimal lines through O.

It is easy to determine the direct relation of the combinations (Γ_1, F_1) and (Γ_2, F_2), that is, the relation in a form independent of the homography H which both define. For this purpose, we observe that (4) may be decomposed into the following three transformations:

$$w = -z, \quad w' = c'\frac{w+3c}{w-c}, \quad z' = -\bar{w}',$$

of which the first represents central symmetry with respect to the origin in the first plane, the second is simply Γ_1, and the third represents symmetry with respect to the axis of ordinates in the second plane. Hence the result:

To any combination (Γ_1, F_1), consisting of a circular transformation and a point, corresponds a second combination (Γ_2, F_2) as follows: the point F_2 is the pole of Γ_1; and Γ_2 is generated by central symmetry with respect to the mid-point of F_1, F_2, followed by Γ_1, followed by symmetry with respect to the mid-line of the points F_1', F_2'; where F_1' is the point into which Γ_1 transforms F_1,

and F_2' is the center of the circle into which Γ_1 transforms the circle passing through F_1 with center F_2. The transformations Γ_1, Γ_2 are opposite in species, and the correspondence considered is mutual.

5. Applications to Invariant Relations. Consider any system of circles passing through a common point O and having their centers on an algebraic curve c_n of the nth order. *Any circular transformation converts such a system into a system of the same kind;* for the centers undergo a projective transformation and hence still lie on a curve c_n' of nth order. In the case $n = 1$, the system is merely a pencil of circles, and the theorem gives a well-known result. For $n = 2$, the theorem announces that a system of circles passing through a common point and with centers lying on a circle, or more generally any conic, is converted into a system with centers lying on a conic.

Any projective relation among the centers of a set of circles of a bundle, corresponds thus to a relation among the circles which is invariant under the group of circular transformations. Consider any number of such circles $\gamma_1, \gamma_2, \cdots$ passing through the point O; let the centers be $P_1, P_2, \cdots$ *Then any projective absolute invariant of the points $P_1, P_2, \cdots$, or more generally of these points together with O, gives an absolute invariant* (with respect to circular transformations) *of the circles $\gamma_1, \gamma_2, \cdots$.*

For example, consider three circles touching each other at a point. The three centers and the point of contact are collinear. The anharmonic ratio of these four points is then an absolute invariant of the three circles. This is perhaps the simplest example of a circular absolute invariant which is not expressible in terms of the various angles of intersection. If we take the more general case of four circles through a point, then the anharmonic ratio of the lines joining say P_1 to P_2, P_3, P_4, O is an absolute invariant of the circles, which can in general be expressed in terms of angles.

COLUMBIA UNIVERSITY, NEW YORK.

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